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$\mathcal{N} = 8$ non-BPS Attractors, Fixed Scalars and Magic Supergravities

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Abstract

We analyze the Hessian matrix of the black hole potential of $\mathcal{N} = 8$, $d = 4$ supergravity, and determine its rank at non-BPS critical points, relating the resulting spectrum to non-BPS solutions (with non-vanishing central charge) of $\mathcal{N} = 2$, $d = 4$ magic supergravities and their "mirror" duals. We find agreement with the known degeneracy splitting of $\mathcal{N} = 2$ non-BPS spectrum of generic special Kähler geometries with cubic holomorphic prepotential. We also relate non-BPS critical points with vanishing central charge in $\mathcal{N} = 2$ magic supergravities to a particular reduction of the $\mathcal{N} = 8$, $\frac{1}{8}$-BPS critical points.

Contents

1 Introduction

After their discovery some time ago [1]-[5], extremal black hole (BH) attractors have been object of intensive study in the last years [6]- [31]. Such a flourishing development mainly can be essentially traced back to new classes of solutions to the attractor equations corresponding to non-BPS horizon geometries.

It has been recently realized that the "effective black hole potential" V_{BH} of $\mathcal{N}\geqslant 2$-extended, $d=4$ supergravities exhibits various species of critical points, whose supersymmetry-preserving and stability features depend on the set of electric and magnetic BH charges.

For what concerns the case $\mathcal{N}=2$, critical points fall into three distinct classes: ($\frac{1}{2}$-) BPS and two non-BPS classes, depending whether the $\mathcal{N}=2$ central charge Z vanishes or not at the BH event horizon. The BPS critical points are known to be always stable (and thus to give rise to actual attractor solutions), as far as they are points at which the metric of the scalar manifold is positive-definite [5].

The stability not guaranteed in the non-BPS cases, in which the Hessian is generally degenerate, *i.e.* it exhibits some "flat" directions. For example, for $\mathcal{N}=2$ supergravities whose vector multiplets' scalar manifold is endowed with special Kähler (SK) d-geometries[1] of complex dimension n_V, it was shown in [10] that the rank of the $2n_V\times 2n_V$ Hessian matrix of V_{BH} (whose real form is the scalar mass matrix) at the non-BPS $Z\neq 0$ critical points has (at most) rank n_V+1 (corresponding to strictly positive eigenvalues), with (at least) n_V-1 "flat" directions (*i.e.* vanishing eigenvalues).

Such a splitting "n_V+1 / n_V-1" of the non-BPS $Z\neq 0$ spectrum has been confirmed in [21], where the $\mathcal{N}=2$ attractor equations were studied in the framework of the homogeneous symmetric SK geometries, which (apart from the case of the irreducible sequence based on quadratic prepotential) are actually particular d-geometries.

In $\mathcal{N}>2$-extended, $d=4$ supergravities the BPS spectrum is degenerate, too. As pointed out in [36], the BPS splitting into non-degenerate (with strictly positive eigenvalues) and "flat" (with vanishing eigenvalues) directions can be explained respectively in terms of the would-be vector multiplets' scalar and hypermultiplets' scalars of the $\mathcal{N}=2$ reduction of the considered $\mathcal{N}>2$ theory. For example, in $\mathcal{N}=8$, $d=4$ supergravity (based on the coset manifold $\frac{E_{7(7)}}{SU(8)}$) the 70×70 Hessian of V_{BH} at the

[1] Following the notation of [32], by d-geometry we mean a SK geometry based on an holomorphic prepotential function of the cubic form $F(X)=d_{ABC}\frac{X^AX^BX^C}{X^0}$ (A, B, $C=0,1,...,n_V$).

(non-degenerate) $\frac{1}{8}$-BPS critical points has rank 30; its 30 strictly positive and 40 vanishing eigenvalues respectively correspond to the 15 vector multiplets and to the 10 hypermultiplets of the $\mathcal{N}=2$, $d=4$ spectrum obtained by reducing $\mathcal{N}=8$ supergravity according to the following branching of the **70** (four-fold antisymmetric) of $SU(8)$:

$$\begin{aligned} & SU(8) \longrightarrow SU(6) \otimes SU(2); \\ & \mathbf{70} \longrightarrow (\mathbf{15},\mathbf{1}) \oplus \left(\overline{\mathbf{15}},\mathbf{1}\right) \oplus (\mathbf{20},\mathbf{2}), \end{aligned} \tag{1.1}$$

where $SU(6)\otimes SU(2)$ is nothing but the symmetry of the 8×8 $\mathcal{N}=8$ central charge matrix Z_{AB} (skew-diagonalizable in the so-called "normal frame" [44]) at the considered non-degenerate $\frac{1}{8}$-BPS critical points. **15**, $\overline{\mathbf{15}}$ and **20** respectively are the two-fold antysimmetric, its complex conjugate and the three-fold antysimmetric of $SU(6)$. In general, the rank of the non-singular $\frac{1}{\mathcal{N}}$-BPS Hessian of V_{BH} in $2\leqslant \mathcal{N}\leqslant 8$-extended, $d=4$ supergravities is [36] $(\mathcal{N}-2)(\mathcal{N}-3)+2n_V$, where n_V stands for the number of matter vector multiplets (for $\mathcal{N}=6$, $n_V=1$ even though there are no vector matter multiplets, because the extra singlet graviphoton counts as a matter field).

The present paper is devoted to the study of the degeneracy of the non-BPS Hessian of V_{BH} in $\mathcal{N}=8$, $d=4$ supergravity, and of the corresponding $\mathcal{N}=2$ theories obtained by consistent truncations. Since such $\mathcal{N}=2$ theories content vector multiplets and hypermultiplets which are some subsets of the kinematical reduction $\mathcal{N}=8\longrightarrow\mathcal{N}=2$ given by Eq. (1.1), the massive and massless modes of the $\mathcal{N}=2$ non-BPS ($Z\neq 0$) Hessian must rearrange following the pattern of degeneracy of the parent $\mathcal{N}=8$ supergravity, when reduced down to $\mathcal{N}=2$ theories.

The plan of the paper is as follows.

In Sect. 2 we review the $\mathcal{N}=2$, $d=4$ *magic* models which can be obtained by consistent reduction of $\mathcal{N}=8$, $d=4$ supergravity. Thence, Sect. 3 deals with the $\mathcal{N}=8$ (non-singular) $\frac{1}{8}$-BPS and non-BPS critical points of V_{BH}; in particular, Subsect. 3.1 reports known results on the $\mathcal{N}=8$, $d=4$ attractor equations and the(symmetries of the)ir solutions, whereas Subsect. 3.2 concerns the Hessian matrix of V_{BH} both at $\frac{1}{8}$-BPS and non-BPS critical points. Thus, in Sect. 4 we consider the $\mathcal{N}=2$ descendants of the $\mathcal{N}=8$, $\frac{1}{8}$-BPS critical points; they divide in $\mathcal{N}=2$, $\frac{1}{2}$-BPS and non-BPS $Z=0$ classes, whose spectra are both studied and compared. In Sect. 5 we perform the same analysis for the descendants of the $\mathcal{N}=8$ non-BPS critical points of V_{BH}, *i.e.* for the $\mathcal{N}=2$ non-BPS $Z\neq 0$ class of critical points of $V_{BH,\mathcal{N}=2}$. We show that the interpretation of the mass degeneracy splitting of $\mathcal{N}=8$ spectra in terms of $\mathcal{N}=2$ multiplets requires a different embedding of the $\mathcal{N}=2$ $\mathcal{R}$-symmetry $SU(2)_H$ in the $\mathcal{R}$-symmetry $SU(8)$ of the parent $\mathcal{N}=8$ theory, depending on the structure and on the eventual supersymmetry-breaking features of the considered class of solutions to attractor equations. Our analysis also yields the interpretation, in terms of the U-duality symmetry $E_{7(7)}$ of $\mathcal{N}=8$, $d=4$ supergravity, of the splitting "n_V+1 / n_V-1" of the $2n_V$ eigenvalues of the $\mathcal{N}=2$ non-BPS $Z\neq 0$ Hessian matrix for generic SK d-geometries of complex dimension n_V, found in [10]. Finally, Sect. 6 contains some general remarks, as well as an outlook of possible future developments.

2 $\mathcal{N}=8$ and $\mathcal{N}=2$ Magic Supergravities

$\mathcal{N}=8$, $d=4$ supergravity is based on the 70-dim. coset $\frac{G}{H}$, where the (continuous) U-duality group G is $E_{7(7)}$ and its maximal compact subgroup (*m.c.s.*) H is $SU(8)$, which is also the (local) $\mathcal{R}$-symmetry of the $\mathcal{N}=8$, $d=4$ supergravity. The vector and hyper multiplets' content of an $\mathcal{N}=2$, $d=4$ reduction of $\mathcal{N}=8$, $d=4$ supergravity is given by a pair

$$(n_V,n_H)\equiv\left(dim_{\mathbb{C}}\left(\frac{G_V}{H_V}\right),dim_{\mathbb{H}}\left(\frac{G_H}{H_H}\right)\right),\ n_V\leqslant 15,\ 2n_H\leqslant 20, \tag{2.1}$$

where $\frac{G_V}{H_V}$ and $\frac{G_H}{H_H}$ respectively stand for the SK vector multiplets' scalar manifold and for the quaternionic Kähler hypermultiplets' scalar manifold. Clearly, in order for the $\mathcal{N}=8\longrightarrow\mathcal{N}=2$ truncation to be consistent, the isometry groups G_V and G_H of the two non-linear σ-models should commute and should be both (proper) subgroups of $G=E_{7(7)}$. We denote $H_V=m.c.s.(G_V)$ and $H_H=m.c.s.(G_H)$. Moreover, H_V always contains a factorized commuting $U(1)$ subgroup, which is promoted to global symmetry (as the Gs) when $n_V=0$; on the other hand, H_H always contains a factorized commuting $SU(2)$

subgroup, which is promoted to global symmetry (as the Gs) when $n_H = 0$. As previously mentioned, $n_V = 15$ and $n_H = 10$ correspond to the reduction (1.1) of $\mathcal{N} = 8$ supergravity, determining two $\mathcal{N} = 2$ supergravities, one based on $\frac{G_V}{H_V} = \frac{SO^*(12)}{SU(6)\otimes U(1)}$ with $(n_V, n_H) = (15, 0)$, and the other one based on $\frac{G_H}{H_H} = \frac{E_{6(2)}}{SU(6)\otimes SU(2)}$ with $(n_V, n_H) = (0, 10)$.

In the following treatment we will consider only $\mathcal{N} = 2$ *maximal* supergravities, *i.e.* $\mathcal{N} = 2$ theories (obtained by consistent truncations of $\mathcal{N} = 8$ supergravity) which cannot be obtained by a further reduction from some other $\mathcal{N} = 2$ theory, which are also *magic*. They are called *magic*, since their symmetry groups are the groups of the famous *Magic Square* of Freudenthal, Rozenfeld and Tits associated with some remarkable geometries [57, 58]. From the analysis performed in [37, 35, 62], only six $\mathcal{N} = 2$, $d = 4$ *maximal magic* supergravities[2] exist which can be obtained by consistently truncating $\mathcal{N} = 8$, $d = 4$ supergravity; they are given[3] by Table 1. The models have been denoted by referring to their SK geometry. $J_3^{\mathbb{H}}$, $J_3^{\mathbb{C}}$ and $J_3^{\mathbb{R}}$ stand for three of the four $\mathcal{N} = 2$, $d = 4$ magic supergravities which, as their 5-dim. versions, are respectively defined by the three simple Jordan algebras $J_3^{\mathbb{H}}$, $J_3^{\mathbb{C}}$ and $J_3^{\mathbb{R}}$ of degree 3 with irreducible norm forms, namely by the Jordan algebras of Hermitian 3×3 matrices over the division algebras of quaternions $\mathbb{H}$, complex numbers $\mathbb{C}$ and real numbers $\mathbb{R}$ [49, 50, 51, 52, 53, 54, 55, 56].

Since $E_{7(-25)}$ is a non-compact form of E_7 (as $E_{7(7)}$ is, as well), the "magic" $\mathcal{N} = 2$, $d = 4$ supergravity defined by the simple Jordan algebra $J_3^{\mathbb{O}}$ over the octonionic division algebra $\mathbb{O}$, having vector multiplets' scalar manifold $\frac{E_{7(-25)}}{E_{6(-78)}\otimes SO(2)}$ ($dim_{\mathbb{C}} = 27$), cannot be obtained from $\mathcal{N} = 8$, $d = 4$ supergravity. Beside the analysis performed in [21], Jordan algebras have been recently connected to extremal black holes also in [61].

"M" subscript denotes the model obtained by performing a $d = 4$ *mirror map* (*i.e.* the composition of two c-maps in $d = 4$) from the original manifold; such an operation maps a model with content (n_V, n_H) to a model with content $(n_H - 1, n_V + 1)$, and thus the mirror of $J_3^{\mathbb{H}}$, with $(n_V, n_H) = (-1, 16)$ and quaternionic manifold $\frac{E_{7(-5)}}{SO(12)\otimes SU(2)}$ does not exist, *at least* in $d = 4$. The *stu* model [47, 48, 23] is *self-mirror*: $stu = stu_M$.

3 $\mathcal{N} = 8$, $d = 4$ Critical Points and Hessian

In Subsect. 3.1 we will review the solutions to the attractor equations of $\mathcal{N} = 8$, $d = 4$ supergravity, mainly following [19] (see [34] for a recent review of Attractor Mechanism in $\mathcal{N} \geqslant 2$-extended, $d = 4$ supergravities). Thence, in Subsect. 3.2we will consider the related critical spectrum given by the Hessian of V_{BH}; while the non-singular $\frac{1}{8}$-case was investigated in [36] (see also [41]), the non-BPS case was hitherto unknown.

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[2] By $E_{7(p)}$ we denote a non-compact form of E_7, where $p \equiv$ (# non-compact − # compact) generators of the group [59, 60]. In such a notation, the compact form of E_7 is $E_{7(-133)}$ ($dim_{\mathbb{R}} E_7 = 133$).

[3] With a slight abuse of language we include among *magic* supergravities the *stu* model, related to the Jordan algebra $\mathbb{R} \oplus \mathbf{\Gamma}_2 = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$, which is the $n = 0$ element of the sequence $\mathbb{R} \oplus \mathbf{\Gamma}_{2+n}$ of reducible Euclidean Jordan algebras of degree 3. $\mathbb{R}$ denotes the one dimensional Jordan algebra and $\mathbf{\Gamma}_{n+2}$ denotes the Jordan algebra of degree 2 associated with a quadratic form of Lorentzian signature (see *e.g.* Table 4 of [21], and Refs. therein).

Due to the group isomorphism $\frac{SO(2,2)}{SO(2)\otimes SO(2)} \sim \left(\frac{SU(1,1)}{U(1)}\right)^2$, the scalar manifold $\frac{G_V}{H_V}$ of the *stu* model, corresponding to the element $n = 0$ of the reducible SK cubic sequence $\frac{SU(1,1)}{U(1)} \otimes \frac{SO(2,2+n)}{SO(2)\otimes SO(2+n)}$ ($n \in \mathbb{N} \cup \{0, -1\}$, $dim_{\mathbb{C}} = n + 3$), is nothing but $\left(\frac{SU(1,1)}{U(1)}\right)^3$.

The image of $\left(\frac{SU(1,1)}{U(1)}\right)^3$ through c-map is given by the 4-dim. (in $\mathbb{H}$) quaternionic manifold $\frac{SO(4,4)}{SO(4)\otimes SO(4)}$, which is the $\frac{G_H}{H_H}$ of the *stu* model. Consistently, it is nothing but the element $n = 0$ of the quaternionic sequence $\frac{SO(4+n,4)}{SO(4+n)\otimes SO(4)}$ ($n \in \mathbb{N} \cup \{0\}$, $dim_{\mathbb{H}} = n + 1$), image of $\frac{SU(1,1)}{U(1)} \otimes \frac{SO(2,2+n)}{SO(2)\otimes SO(2+n)}$ through c-map (see *e.g.* Table 4 of [38], and [39]).

Finally, the 1-dim. (in $\mathbb{H}$) quaternionic manifold $\frac{SU(2,1)}{SU(2)\otimes U(1)}$, corresponding to the $\frac{G_H}{H_H}$ of the model $J_3^{\mathbb{H}}$, is the so-called *universal hypermultiplet*, given by the c-map of the case $n_V = 0$, *i.e.* of *pure* $\mathcal{N} = 2$, $d = 4$ supergravity, which (among the homogeneous SK geometries) is defined as the *n = 0 limit* of the rank-1 sequence of quadratic irreducible SK manifolds $\frac{SU(1,n)}{U(1)\otimes SU(n)}$ ($n \in \mathbb{N}$, $dim_{\mathbb{C}} = n$) [40].

	G_V	G_H	H_V	H_H	$\frac{G_V}{H_V} \otimes \frac{G_H}{H_H}$	(n_V, n_H)
$J_3^{\mathbb{H}}$	$SO^*(12)$	$SU(2)$	$SU(6) \otimes U(1)$	$-$	$\frac{SO^*(12)}{SU(6)\otimes U(1)}$	$(15,0)$
$J_3^{\mathbb{C}}$	$SU(3,3)$	$SU(2,1)$	$SU(3) \otimes SU(3) \otimes U(1)$	$SU(2) \otimes U(1)$	$\frac{SU(3,3)}{SU(3)\otimes SU(3)\otimes U(1)} \otimes \frac{SU(2,1)}{SU(2)\otimes U(1)}$	$(9,1)$
$J_3^{\mathbb{R}}$	$Sp(6,\mathbb{R})$	$G_{2(2)}$	$SU(3) \otimes U(1)$	$SU(2) \otimes SU(2)$	$\frac{Sp(6,\mathbb{R})}{SU(3)\otimes U(1)} \otimes \frac{G_{2(2)}}{SO(4)}$	$(6,2)$
stu	$SU(1,1) \otimes SO(2,2)$	$SO(4,4)$	$U(1) \otimes SO(2) \otimes SO(2)$	$SO(4) \otimes SO(4)$	$\frac{SU(1,1)}{U(1)} \otimes \frac{SO(2,2)}{SO(2)\otimes SO(2)} \otimes \frac{SO(4,4)}{SO(4)\otimes SO(4)}$	$(3,4)$
$J_{3,M}^{\mathbb{R}}$	$SU(1,1)$	$F_{4(4)}$	$U(1)$	$USp(6) \otimes SU(2)$	$\frac{SU(1,1)}{U(1)} \otimes \frac{F_{4(4)}}{USp(6)\otimes SU(2)}$	$(1,7)$
$J_{3,M}^{\mathbb{C}}$	$U(1)$	$E_{6(2)}$	$-$	$SU(6) \otimes SU(2)$	$\frac{E_{6(2)}}{SU(6)\otimes SU(2)}$	$(0,10)$

Table 1: **Data of the *magic* $\mathcal{N}=2$, $d=4$ supergravities obtained as consistent truncation of ($\frac{G}{H}=\frac{E_{7(7)}}{SU(8)}$-based) $\mathcal{N}=8$, $d=4$ supergravity**

3.1 Solutions to Attractor Equations

The black hole potential of $\mathcal{N}=8$, $d=4$ supergravity (based on the real coset $\frac{E_{7(7)}}{SU(8)}$) [42] reads as follows [43, 5] ($A, B = 1, \dots, 8$ throughout):

$$V_{BH} = \frac{1}{2} Z_{AB} \overline{Z}^{AB}, \tag{3.1}$$

where Z_{AB} (and its complex conjugate $\overline{Z}^{AB}$) is the central charge matrix (and its conjugate), sitting in the two-fold antisymmetric complex **28** of $E_{7(7)}$. It depends on 70 $\left(= dim_{\mathbb{R}}\left(\frac{E_{7(7)}}{SU(8)}\right)\right)$ real scalars ϕ^i ($i=1,\dots,70$ throughout, unless otherwise noted), where the local $SU(8)$ symmetry was used to remove 63 scalars from the representation **133** of scalars in $E_{7(7)}$.

The $SU(8)$-covariant derivatives [43] of the central charge matrix are defined by the *Maurer-Cartan equations* for $\frac{E_{7(7)}}{SU(8)}$:

$$D_i Z_{AB} = \frac{1}{2} \overline{Z}^{CD} P_{ABCD,i} \Leftrightarrow D_i \overline{Z}^{AB} = \frac{1}{2} Z_{CD} \overline{P}^{ABCD}_{,i}, \tag{3.2}$$

where $P_{ABCD} = P_{i,[ABCD]}d\phi^i$ is the 70×70 vielbein 1-form of $\frac{E_{7(7)}}{SU(8)}$, sitting in the **70** (four-fold antisymmetric) of the stabylizer $SU(8)$, and satisfying to the *self-dual reality* condition

$$\overline{P}^{ABCD} = \frac{1}{4!}\epsilon^{ABCDEFGH}P_{EFGH} \Leftrightarrow P_{ABCD} = \frac{1}{4!}\epsilon_{ABCDEFGH}\overline{P}^{EFGH}, \tag{3.3}$$

$\epsilon_{ABCDEFGH}$ being the rank-8 completely antisymmetric Ricci-Levi-Civita tensor of $SU(8)$. By using Eqs. (3.2) and (3.3), and by exploiting the invertibility (non-singularity) of $P_{ABCD,i}$, the criticality conditions for V_{BH} can be rewritten as [43, 5, 19]

$$\overline{Z}^{[AB}\overline{Z}^{CD]} + \frac{1}{4!}\epsilon^{ABCDEFGH}Z_{[EF}Z_{GH]} = 0, \tag{3.4}$$

which are usually referred to as the $\mathcal{N} = 8$, $d = 4$ attractor equations. They are purely algebraic in the $\left(Z_{AB}, \overline{Z}^{AB}\right)$, and they hold for all non-singular (*i.e.* with $V_{BH} \neq 0$) critical points of V_{BH} in $\frac{E_{7(7)}}{SU(8)}$ at which $P_{ABCD,i}$ is invertible.

The local $SU(8)$ symmetry allows one to go to the so-called "normal frame" [44]. In such a frame, Z_{AB} and the unique Cartan-Cremmer-Julia quartic invariant J_4 [45, 42] of the fundamental representation **56** of $E_{7(7)}$ respectively read as follows ($\epsilon \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the 2-dim. symplectic metric):

$$\begin{aligned}
&Z_{AB,normal} = \begin{pmatrix} z_1\epsilon & 0 & 0 & 0 \\ 0 & z_2\epsilon & 0 & 0 \\ 0 & 0 & z_3\epsilon & 0 \\ 0 & 0 & 0 & z_4\epsilon \end{pmatrix} \equiv \begin{pmatrix} \rho_1\epsilon & 0 & 0 & 0 \\ 0 & \rho_2\epsilon & 0 & 0 \\ 0 & 0 & \rho_3\epsilon & 0 \\ 0 & 0 & 0 & \rho_4\epsilon \end{pmatrix} e^{i\varphi/4}; \\
&z_i \equiv \rho_i e^{i\varphi/4} \in \mathbb{C},\ \rho_i \in \mathbb{R}^+,\ i = 1, 2, 3, 4, \\
&\rho_1 \geqslant \rho_2 \geqslant \rho_3 \geqslant \rho_4 \geqslant 0,\ \varphi \in [0, 8\pi)\,.
\end{aligned} \tag{3.5}$$

$$J_{4,normal} = \left[(\rho_1 + \rho_2)^2 - (\rho_3 + \rho_4)^2\right]\left[(\rho_1 - \rho_2)^2 - (\rho_3 - \rho_4)^2\right] + 8\rho_1\rho_2\rho_3\rho_4(cos\varphi - 1). \tag{3.6}$$

Note that $Z_{AB,normal}$ has an $(SU(2))^4$ symmetry. The $\mathcal{N} = 8$ attractor equations (3.4) acquire the following simple form [19]:

$$\begin{cases} z_1z_2 + \overline{z_3z_4} = 0; \\ z_1z_3 + \overline{z_2z_4} = 0; \\ z_2z_3 + \overline{z_1z_4} = 0. \end{cases} \tag{3.7}$$

As expected from the analysis of [46, 33], $\mathcal{N} = 8$, $d = 4$ extremal black hole attractor equations (3.7) have only 2 distinct classes of non-singular solutions ($\frac{1}{8}$-BPS for $J_4 > 0$, non-BPS for $J_4 < 0$):

1. **$\frac{1}{8}$-BPS:**

$$\rho_1 = \rho_{\frac{1}{8}-BPS} \in \mathbb{R}_0^+,\ \varphi_{\frac{1}{8}-BPS} \in [0, 8\pi)\,,\ \rho_{2,\frac{1}{8}-BPS} = \rho_{3,\frac{1}{8}-BPS} = \rho_{4,\frac{1}{8}-BPS} = 0. \tag{3.8}$$

The corresponding orbit of supporting BH charges in the **56** of $E_{7(7)}$ is $\mathcal{O}_{\frac{1}{8}-BPS} = \frac{E_{7(7)}}{E_{6(2)}}$, with $J_{4,normal,\frac{1}{8}-BPS} = \rho^4_{\frac{1}{8}-BPS} > 0$ and classical entropy $S_{BH,\frac{1}{8}-BPS} = \pi\sqrt{J_{4,normal,\frac{1}{8}-BPS}} = \pi\rho^2_{\frac{1}{8}-BPS}$. As implied by Eq. (3.8), $Z_{AB,normal,\frac{1}{8}-BPS}$ has symmetry enhancement $(SU(2))^4 \longrightarrow SU(6) \otimes SU(2) = m.c.s.\left(E_{6(2)}\right)$. Notice that $\varphi_{\frac{1}{8}-BPS}$ is actually undetermined.

2. **non-BPS:**

$$\rho_{1,non-BPS} = \rho_{2,non-BPS} = \rho_{3,non-BPS} = \rho_{4,non-BPS} = \rho_{non-BPS} \in \mathbb{R}_0^+,\quad \varphi_{non-BPS} = \pi. \tag{3.9}$$

The corresponding orbit of supporting BH charges in the **56** of $E_{7(7)}$ is $\mathcal{O}_{non-BPS} = \frac{E_{7(7)}}{E_{6(6)}}$, with $J_{4,normal,non-BPS} = -16\rho^4_{non-BPS} < 0$ and classical entropy $S_{BH,non-BPS} = \pi\sqrt{-J_{4,normal,non-BPS}} = 4\pi\rho^2_{non-BPS}$. The deep meaning of the extra factor 4 in $S_{BH,non-BPS}$ as compared to $S_{BH,\frac{1}{8}-BPS}$ can be clearly explained when considering the so-called "*stu* interpretation" of $\mathcal{N} = 8$ regular critical points

[19]. As implied by Eq. (3.9), $Z_{AB,normal,non-BPS}$ has symmetry enhancement $(SU(2))^4 \longrightarrow USp(8) = m.c.s.\left(E_{6(6)}\right)$; indeed

$$Z_{AB,normal,non-BPS} = e^{i\frac{\pi}{4}}\rho_{non-BPS}\Omega_{AB}, \tag{3.10}$$

where Ω_{AB} is the $USp(8)$ metric:

$$\Omega_{AB} \equiv \begin{pmatrix} \epsilon & & & \\ & \epsilon & & \\ & & \epsilon & \\ & & & \epsilon \end{pmatrix}. \tag{3.11}$$

Thus, as pointed out at the end of the Introduction of [21], the symmetry of $Z_{AB,normal}$ gets enhanced at the particular points of $\frac{E_{7(7)}}{SU(8)}$ given by the non-singular solutions of $\mathcal{N} = 8$, $d = 4$ attractor equations (3.7). In general, *the invariance properties of the non-singular solutions to attractor eqs. are given by the m.c.s. of the stabilizer of the corresponding supporting BH charge orbit.*

3.2 Critical Spectra

Let us now consider the Hessian of V_{BH}. By further covariantly differentiating V_{BH}, one gets [36]

$$H_{ij} \equiv D_i D_j V_{BH} = \frac{1}{2} Z_{CD}\overline{Z}^{AB}\overline{P}^{CDEF}_{,j} P_{ABEF,i} = H_{ji}. \tag{3.12}$$

1. $\frac{1}{8}$-BPS:

By recalling Eq. (3.8), it can be computed that ($a, b = 3, ..., 8$) [36]

$$\begin{aligned} H_{ij,\frac{1}{8}-BPS} &= \tfrac{1}{2}\left[Z_{CD}\overline{Z}^{AB}\overline{P}^{CDEF}_{,j} P_{ABEF,i}\right]_{\frac{1}{8}-BPS} = \\ &= 2\rho^2_{\frac{1}{8}-BPS}\left[\overline{P}^{12ab}_{,j} P_{12ab,i}\right]_{\frac{1}{8}-BPS} = \tfrac{1}{12}\rho^2_{\frac{1}{8}-BPS}\epsilon^{12abEFGH}\left[P_{EFGH,j}P_{12ab,i}\right]_{\frac{1}{8}-BPS}. \end{aligned} \tag{3.13}$$

As observed in [36], the pattern of degeneracy of the modes of $H_{ij,\frac{1}{8}-BPS}$ can be understood by noticing that the very structure of the non-singular $\frac{1}{8}$-BPS solution (3.8), in which only one eigenvalue of the skew-diagonal matrix $Z_{AB,normal}$ is not vanishing, yields that the $\mathcal{N} = 8$ theory effectively reduces to an $\mathcal{N} = 2$ theory. Consequently, the degeneracy splitting of the eigenvalues of $H_{ij,\frac{1}{8}-BPS}$ will respect the multiplicity of the $\mathcal{N} = 2$ scalar degrees of freedom: the "flat" directions will correspond to the $\mathcal{N} = 2$ hypermultiplet content, whereas the "non-flat" directions (with strictly positive eigenvalues) will correspond to the $\mathcal{N} = 2$ vector multiplet content.

The crucial point is the choice of the kinematical reduction $\mathcal{N} = 8 \longrightarrow \mathcal{N} = 2$. As previously mentioned, in the non-singular $\frac{1}{8}$-BPS case it is performed through the branching of **70** of $SU(8)$ along the $\frac{1}{8}$-BPS enhanced symmetry $SU(6) \otimes SU(2)$ given by Eq. (1.1), yielding:

i) $2n_V = 30$ strictly positive directions (*massive Hessian modes*), corresponding to 15 complex $\mathcal{N} = 2$ vector multiplets' scalars, sitting into the $(\mathbf{15}, \mathbf{1}) \oplus \left(\overline{\mathbf{15}}, \mathbf{1}\right)$ of $SU(6) \otimes SU(2)$, and parameterized by the 30 real components P_{abcd};

and

ii) $4n_H = 40$ "flat" directions (*massless Hessian modes*), corresponding to 10 quaternionic $\mathcal{N} = 2$ hypermultiplets' scalars, sitting into the $(\mathbf{20}, \mathbf{2})$ of $SU(6) \otimes SU(2)$, and parameterized by the 40 real components[4] $\{P_{1abc}, P_{2abc}\}$.

Thus, at $\mathcal{N} = 2$, $\frac{1}{2}$-BPS critical points of $V_{BH,\mathcal{N}=2}$ Eq. (1.1) can be written as follows:

$$\mathbf{70} \longrightarrow \underbrace{\overbrace{(\mathbf{15},\mathbf{1})}^{m\neq 0} \oplus \overbrace{(\overline{\mathbf{15}},\mathbf{1})}^{m\neq 0}}_{\text{vectors' scalars}} \oplus \underbrace{\overbrace{(\mathbf{20},\mathbf{2})}^{m=0}}_{\text{hypers' scalars}}, \tag{3.14}$$

Under the branching (1.1) P_{ABCD} decomposes as $P_{ABCD} \longrightarrow \{P_{1abc}, P_{2abc}, P_{abcd}\}$. As it holds true in general (also at non-BPS non-singular critical points), the $\mathcal{N} = 2$ vector and hyper scalar degrees of

[4] Notice that, due to the self-dual reality condition (3.3), P_{12ab} can be re-expressed in terms of the other independent component of P_{ABCD}.

freedom are respectively singlets and doublets of the $\mathcal{N}=2$ $\mathcal{R}$-symmetry $SU(2)_{\mathcal{R},\mathcal{N}=2} \equiv SU(2)_H$, which in general lies inside the whole $\mathcal{N}=8$ $\mathcal{R}$-symmetry $SU(8)$.

Thus, in the non-singular $\mathcal{N}=8$, $\frac{1}{8}$-BPS case *all* $\mathcal{N}=2$ vector multiplets' scalar degrees of freedom of H_{ij} are massive, while *all* its $\mathcal{N}=2$ hypermultiplets' scalar degrees of freedom are massless; this can be understood by observing that the preservation of 4 supersymmetric degrees of freedom forces such two different kind of $\mathcal{N}=2$ degrees of freedom to follow separated mass degeneracy patterns.

2. non-BPS:

The same can be intuitively guessed *not* to hold in the (non-singular) non-BPS case, where no supersymmetric degrees of freedom are preserved by the critical solution. In fact, what actually happens is that, for what concerns the mass degeneracy spliiting, the $\mathcal{N}=2$ vector and hyper scalar degrees of freedom of H_{ij} mix together, in a way which follows the various possibilities yielded by *all* the *maximal magic* $\mathcal{N}=2$, $d=4$ supergravities which are consistent truncations of $\mathcal{N}=8$, $d=4$ supergravity (given by Table 1).

Indeed, by recalling Eqs. (3.9) and (3.10), it can be computed that

$$\begin{aligned} H_{ij,non-BPS} &= \tfrac{1}{2}\left[Z_{CD}\overline{Z}^{AB}\overline{P}^{CDEF}_{,j}P_{ABEF,i}\right]_{non-BPS} = \\ &= \tfrac{1}{2}\rho^2_{non-BPS}\left[\begin{array}{l} \frac{4}{27}\epsilon^{ABCDEFGH}P_{[ABCD|,i}P_{|EFGH],j}+ \\ \\ +\left(32-\frac{1}{18}\right)P_{ABCD,i}P_{EFGH,j}\Omega^{[AB}\Omega^{CD]}\Omega^{[EF}\Omega^{GH]} \end{array}\right]_{non-BPS}. \end{aligned} \tag{3.15}$$

In this case, the relevant branching of the **70** of the stabylizer $SU(8)$ is along the non-BPS enhanced symmetry $USp(8)$:

$$\begin{aligned} &SU(8) \longrightarrow USp(8); \\ &\mathbf{70} \longrightarrow \mathbf{42} \oplus \mathbf{27} \oplus \mathbf{1}, \end{aligned} \tag{3.16}$$

where **42**, **27** and **1** respectively are the four-fold antysimmetric (traceless), two-fold antysimmetric (traceless) and the singlet of $USp(8)$. Under the branching (3.16) P_{ABCD} decomposes as follows:

$$P_{ABCD} \longrightarrow \left\{\hat{P}_{ABCD}, \hat{P}_{AB}, \hat{P}^0\right\};$$

$$\left\{\begin{array}{l} \mathbf{1} \text{ of } USp(8): \hat{P}^0 \equiv \frac{1}{2^4}P_{ABCD}\Omega^{[AB}\Omega^{CD]}; \\ \mathbf{27} \text{ of } USp(8): \hat{P}_{AB} \equiv \frac{3}{2}P_{ABCD}\Omega^{CD} - 3\hat{P}^0\Omega_{AB},\ \hat{P}_{AB} = \hat{P}_{[AB]},\ \hat{P}_{AB}\Omega^{AB} = 0; \\ \mathbf{42} \text{ of } USp(8): \hat{P}_{ABCD} \equiv P_{ABCD} - \hat{P}_{[AB}\Omega_{CD]} - \hat{P}^0\Omega_{[AB}\Omega_{CD]},\ \hat{P}_{ABCD} = \hat{P}_{[ABCD]},\ \hat{P}_{ABCD}\Omega^{CD} = 0. \end{array}\right. \tag{3.17}$$

By using such an $USp(8)$-covariant decomposition of P_{ABCD}, the result (3.15) can be rewritten as follows:

$$H_{ij,non-BPS} = \frac{1}{2}\rho^2_{non-BPS}\left[\left(\frac{2}{3}\right)^4\overline{\hat{P}}^{AB}_{,j}\hat{P}_{AB,i} + 2^{13}\hat{P}^0_{,i}\hat{P}^0_{,j}\right]_{non-BPS}, \tag{3.18}$$

where the barred quantities have definitions and properties analogue to the ones in Eq. (3.17), to which they are related by the self-dual reality condition (3.3), too.

Thus, one sees that the non-BPS kinematical reduction $\mathcal{N}=8 \longrightarrow \mathcal{N}=2$ performed through the branching of **70** of $SU(8)$ along the non-BPS enhanced symmetry$USp(8)$ given by Eq. ((3.16)) yields a different mass degeneracy splitting with respect to the $\frac{1}{8}$-BPS case treated above. Indeed, as evident from Eq. (3.18), $H_{ij,non-BPS}$ is splitted in:

i) 28 strictly positive directions (*massive Hessian modes*), sitting into the $\mathbf{27} \oplus \mathbf{1}$ of $USp(8)$, and parameterized by the $27+1$ real components $\hat{P}_{AB}$ and $\hat{P}^0$;

and

ii) 42 "flat" directions (*massless Hessian modes*), sitting into the **42** of $USp(8)$, and parameterized by the 42 real components $\hat{P}_{ABCD}$.

Thus, at $\mathcal{N}=8$ non-BPS critical points of V_{BH} Eq. (3.16) can be written as follows:

$$\mathbf{70} \longrightarrow \overbrace{\mathbf{42}}^{m=0} \oplus \overbrace{\mathbf{27}}^{m\neq 0} \oplus \overbrace{\mathbf{1}}^{m\neq 0}. \tag{3.19}$$

As we will see below, the identification of the massive and massless Hessian modes with the $\mathcal{N}=2$ vector multiplets' and hypermultiplets' scalars is model-dependent.

However, from the splitting "n_V+1 / n_V-1" found in [10] (holding for generic SK d-geometries), we can state the following result for non-BPS $Z\neq 0$ critical points of all $\mathcal{N}=2$, $d=4$ supergravities listed in Table 1: given a pair (n_V, n_H) describing the multiplets' content of the model, $4n_H+n_V-1$ massless real modes sit in the **42** of $USp(8)$, while n_V real massive modes sit in the **27** of $USp(8)$ (the remaining 1 real massive mode sitting in the singlet **1** of $USp(8)$).

4 $\mathcal{N}=8$, $\frac{1}{8}$-BPS Critical Points and their $\mathcal{N}=2$ Descendants

As pointed out above, $\mathcal{N}=8$, $\frac{1}{8}$-BPS critical points of V_{BH} have symmetry $SU(6)\otimes SU(2)_{\mathcal{R}}$, where $SU(2)_{\mathcal{R}}$ is the $SU(2)$ factor of the $\mathcal{N}=8$ $\mathcal{R}$-symmetry $SU(8)$ which commutes with $SU(6)$. The 70×70 $\frac{1}{8}$-BPS Hessian matrix $H_{ij,\frac{1}{8}-BPS}$ of V_{BH} has rank 30, corresponding to the $\mathcal{N}=8\longrightarrow\mathcal{N}=2$ kinematical decomposition (1.1). It is worth noticing that, under the same branching, the **56** fundamental representation of the $\mathcal{N}=8$ U-duality group $G=E_{7(7)}$ decomposes into representation of the $\frac{1}{8}$-BPS symmetry $SU(6)\otimes SU(2)_{\mathcal{R}}$ as follows:

$$\mathbf{56}\longrightarrow(\mathbf{15},\mathbf{1})\oplus\left(\overline{\mathbf{15}},\mathbf{1}\right)\oplus(\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{1})\oplus(\mathbf{6},\mathbf{2})\oplus\left(\overline{\mathbf{6}},\mathbf{2}\right), \tag{4.1}$$

which consistently gives 16 electric and 16 magnetic charges for the $15+1$ Abelian vectors of the $\mathcal{N}=2$ matter and gravity supermultiplets. The remaining charges from the decomposition (4.1) pertain to the graviphotons which are partners of the 6 remaining gravitino multiplets $6\left(\frac{3}{2},2\left(1\right),\frac{1}{2}\right)$ in the $\mathcal{N}=8\longrightarrow\mathcal{N}=2$ reduction (1.1), which precisely have $(\mathbf{6},\mathbf{2})\oplus\left(\overline{\mathbf{6}},\mathbf{2}\right)$ electric and magnetic field strenghts.

4.1 $\mathcal{N}=2$, $\frac{1}{2}$-BPS

For the $\mathcal{N}=2$, $d=4$ supergravities listed in Table 1, the enhanced symmetry $\mathcal{S}_{\frac{1}{2}-BPS}$ of $\mathcal{N}=2$, $d=4$ $\frac{1}{2}$-BPS critical points of $V_{BH,\mathcal{N}=2}$ is given by [36, 21]

$$\mathcal{S}_{\frac{1}{2}-BPS}=H_0\otimes H_H, \tag{4.2}$$

where H_0 is the stabylizer of the $\mathcal{N}=2$, $\frac{1}{2}$-BPS-supporting BH charge orbit[5], and H_H is the stabylizer of $\frac{G_H}{H_H}$. Furthermore, $\mathcal{N}=2$, $\frac{1}{2}$-BPS case has $\mathcal{N}=2$ quartic G_V-invariant $I_4>0$, where I_4 is nothing but a suitable "truncation" of the $E_{7(7)}$-invariant J_4. Since the sign of the U-duality group invariant (built out from the symplectic representation of the U-duality group) does not change in the $\mathcal{N}=8\longrightarrow\mathcal{N}=2$ supersymmetry reduction, it is clear that the $\mathcal{N}=2$, $\frac{1}{2}$-BPS case comes from the reduction of the $\mathcal{N}=8$, $\frac{1}{8}$-BPS case.

Thus, $\mathcal{S}_{\frac{1}{2}-BPS}$ must be included in the overall enhanced symmetry $SU(6)\otimes SU(2)_{\mathcal{R}}$ of the $\mathcal{N}=8$, $\frac{1}{8}$-BPS case:

$$\mathcal{S}_{\frac{1}{2}-BPS}\subseteq SU(6)\otimes SU(2)_{\mathcal{R}}. \tag{4.3}$$

The very structure of the quaternionic Kähler manifold $\frac{G_H}{H_H}$ yields that H_H always include at least one explicit factor $SU(2)$, which is promoted to a global symmetry in the case $n_H=0$. Thus, H_H can always (for $n_H\neq 0$) be rewritten as

$$H_H=\frac{H_H}{SU(2)}\otimes SU(2). \tag{4.4}$$

In general, the $\mathcal{N}=2$ $\mathcal{R}$-symmetry group $SU(2)_{\mathcal{R},\mathcal{N}=2}$ is identified with the $SU(2)$ factorized in the r.h.s. of Eq. (4.4), which in the follow we will denote with the subscript "H":

$$SU(2)_{\mathcal{R},\mathcal{N}=2}=SU(2)_H\subseteq H_H. \tag{4.5}$$

[5]Here and in the following treatment we will make use of the notation set up in [21]. H_0 is defined (for $n_V\neq 0$) as $H_0\equiv\frac{H_V}{U(1)}$ [21].

	$\frac{1}{2}$-BPS orbit $\mathcal{O}_{\frac{1}{2}-BPS}=\frac{G_V}{H_0}$	$H_0\equiv\frac{H_V}{U(1)}$	$\frac{H_H}{SU(2)_{\mathcal{R}}=SU(2)_H}$
$J_3^{\mathbb{H}}$	$\frac{SO^*(12)}{SU(6)}$	$SU(6)$	$\nexists H_H,\quad SU_H(2)=SU(2)_{\mathcal{R}}=G_H$
$J_3^{\mathbb{C}}$	$\frac{SU(3,3)}{SU(3)\otimes SU(3)}$	$SU(3)\otimes SU(3)$	$U(1)$
$J_3^{\mathbb{R}}$	$\frac{Sp(6,\mathbb{R})}{SU(3)}$	$SU(3)$	$SU(2)$
stu	$\frac{(SU(1,1))^3}{(U(1))^2}$	$(U(1))^2$	$(SU(2))^3$
$J_{3,M}^{\mathbb{R}}$	$SU(1,1)$	$\mathbb{I}$	$USp(6)$
$J_{3,M}^{\mathbb{C}}$	$-$	$-$	$SU(6)$

Table 2: **The $\frac{1}{2}$-BPS supporting BH charge orbit $\mathcal{O}_{\frac{1}{2}-BPS}$, and the compact groups H_0 and $\frac{H_H}{SU(2)_{\mathcal{R}}}$ (relevant at $\mathcal{N}=2$, $\frac{1}{2}$-BPS critical points) for the $\mathcal{N}=2$, $d=4$ supergravities listed in Table 1**

The identification determining the $\mathcal{N}=2$, $\frac{1}{2}$-BPS case as descendant of the $\mathcal{N}=8$, $\frac{1}{8}$-BPS case reads as follows (recall Eq. (3.8)):

$$Z_{12,\frac{1}{8}-BPS}\equiv z_{1,\frac{1}{8}-BPS}=e^{i\varphi/4}\rho_{\frac{1}{8}-BPS}=Z_{\frac{1}{2}-BPS}\in\mathbb{C}_0. \tag{4.6}$$

Therefore, at $\mathcal{N}=2$, $\frac{1}{2}$-BPS critical points of $V_{BH,\mathcal{N}=2}$ (which preserve 4 supersymmetry charges, and are always stable [5], thus corresponding to attractor configurations), the $\mathcal{N}=8\longrightarrow\mathcal{N}=2$ kinematical decomposition (1.1) identifies $SU(2)_{\mathcal{R}}$ on the r.h.s. of Eq. (4.3) with the $\mathcal{N}=2$ $\mathcal{R}$-symmetry $SU(2)_H$:

$$SU(2)_{\mathcal{R}}=SU(2)_H. \tag{4.7}$$

Thus, Eq. (4.4) can be rewritten as

$$H_H=\frac{H_H}{SU(2)_{\mathcal{R}}}\otimes SU(2)_{\mathcal{R}}, \tag{4.8}$$

which, by Eq. (4.3), implies that

$$H_0\otimes\frac{H_H}{SU(2)_{\mathcal{R}}}\subseteq SU(6). \tag{4.9}$$

The corresponding data for all the $\mathcal{N}=2$, $d=4$ supergravities which are consistent truncations of the $\mathcal{N}=8$, $d=4$ theory (listed in Table 1) are given in Table 2 (for the columns "$\mathcal{O}_{\frac{1}{2}-BPS}$" and "$H_0$" refer to Tables 3 and 8 of [21]).

From Table 2 it is also evident that $SU(2)_{\mathcal{R}}$ has necessarily to be chosen in H_H, because in all models H_0 does not contain a factorized $SU(2)$. Moreover, two orders of considerations follow:

i) $H_0\otimes\frac{H_H}{SU(2)_{\mathcal{R}}}$ is a *proper* subgroup of $SU(6)$ in all models but the two limit models $J_3^{\mathbb{H}}$ (having $n_H=0$, and thus H_H undefined) and $J_{3,M}^{\mathbb{C}}$ (having $n_V=0$, and thus H_0 undefined and corresponding to a Reissner-Nördstrom extremal BH, only having $\frac{1}{2}$-BPS critical points).

For $J_3^{\mathbb{H}}$, $SU(2)_{\mathcal{R}} = SU(2)_H$ is identified with the global symmetry $SU(2) = G_H$ due to $n_H = 0$.

On the other hand, for $J_{3,M}^{\mathbb{C}}$ it holds that $\mathcal{S}_{\frac{1}{2}-BPS} = H_H = SU(6) \otimes SU(2)_{\mathcal{R}}$, *i.e.* the enhanced $\mathcal{N} = 2$, $\frac{1}{2}$-BPS symmetry $\mathcal{S}_{\frac{1}{2}-BPS}$, the stabylizer of the quaternionic Kähler manifold $\frac{G_H}{H_H}$ and the enhanced $\mathcal{N} = 8$, $\frac{1}{8}$-BPS symmetry coincide.

ii) Two models exist where an *apriori* arbitrariness in the identification of $SU(2)_H$ in H_H exists: $J_3^{\mathbb{R}}$ and *stu*.

However, in $J_3^{\mathbb{R}}$ such an arbitrariness is removed by the quantum numbers of the hypermultiplets' scalars (which are always doublets of $SU(2)_H$); the "right" $SU(2)$ to choose is the one promoted to a global symmetry in the limit case $n_H = 0$. On the other hand, in *stu* case the arbitrariness of choice is removed by the noteworthy *triality symmetry* of the model.

4.2 $\mathcal{N} = 2$ non-BPS $Z = 0$

For the $\mathcal{N} = 2$, $d = 4$ supergravities listed in Table 1, the overall symmetry $\mathcal{S}_{non-BPS,Z=0}$ of $\mathcal{N} = 2$, $d = 4$ non-BPS $Z = 0$ critical points of $V_{BH,\mathcal{N}=2}$ is given by [21]

$$\mathcal{S}_{non-BPS,Z=0} = \widetilde{h}' \otimes H_H, \tag{4.10}$$

where $\widetilde{h}'$ is the *m.c.s.* (factorized by $U(1)$) of the stabylizer $\widetilde{H}$ of the $\mathcal{N} = 2$ non-BPS $Z = 0$-supporting BH charge orbit [21]. Furthermore, $\mathcal{N} = 2$ non-BPS $Z = 0$ case has $\mathcal{N} = 2$ quartic G_V-invariant $I_4 > 0$, as the $\mathcal{N} = 2$, $\frac{1}{2}$-BPS case. Thus, it is clear that $\mathcal{N} = 2$ non-BPS $Z = 0$ case comes from the very same $\mathcal{N} = 8 \longrightarrow \mathcal{N} = 2$ supersymmetry reduction giving raise to $\mathcal{N} = 2$, $\frac{1}{2}$-BPS case. Thus, $\mathcal{S}_{non-BPS,Z=0}$ must be included in the overall enhanced symmetry $SU(6) \otimes SU(2)_{\mathcal{R}}$ of the $\mathcal{N} = 8$, $\frac{1}{8}$-BPS case:

$$\mathcal{S}_{non-BPS,Z=0} \subseteq SU(6) \otimes SU(2)_{\mathcal{R}}. \tag{4.11}$$

The identification determining the $\mathcal{N} = 2$ non-BPS $Z = 0$ case as descendant of the $\mathcal{N} = 8$, $\frac{1}{8}$-BPS case reads as follows (recall that $Z_{non-BPS,Z=0} = 0$):

$$Z_{12,\frac{1}{8}-BPS} \equiv z_{1,\frac{1}{8}-BPS} = e^{i\varphi/4}\rho_{\frac{1}{8}-BPS} = (D_i Z)_{non-BPS,Z=0} \neq 0, \tag{4.12}$$

where i is one particular element of the set $\{1, ..., n_V\}$. In this sense, the key difference with respect to the previously treated $\mathcal{N} = 2$, $\frac{1}{2}$-BPS case is that the $\mathcal{N} = 2$ central charge is interchanged with one $\mathcal{N} = 2$ *matter charge*.

This leads to the fact that for $\mathcal{N} = 2$ models under consideration which exhibit "flat" Hessian directions at $\mathcal{N} = 2$ non-BPS $Z = 0$ critical points of $V_{BH,\mathcal{N}=2}$ (namely $J_3^{\mathbb{H}}$, $J_3^{\mathbb{C}}$ and $J_3^{\mathbb{R}}$) the $SU(2)_{\mathcal{R}}$ of the enhanced $\mathcal{N} = 8$, $\frac{1}{8}$-BPS symmetry $SU(2)_{\mathcal{R}} \otimes SU(6)$ is not identified with the $SU(2)_{\mathcal{R},\mathcal{N}=2}$ (*i.e.* with (one of) the $SU(2)$(s) factorized in H_H) any more, but rather it is identified with an explicit $SU(2)$ factor in $\widetilde{h}'$. Thus, for these models $\widetilde{h}'$ can be rewritten as

$$J_3^{\mathbb{H}}, J_3^{\mathbb{C}}, J_3^{\mathbb{R}} : \widetilde{h}' = \frac{\widetilde{h}'}{SU(2)} \otimes SU(2). \tag{4.13}$$

By making the identification $SU(2)_{\mathcal{R}} = SU(2)$ factor on the r.h.s. of Eq. (4.13), one can thus rewrite Eq. (4.11) as follows:

$$J_3^{\mathbb{H}}, J_3^{\mathbb{C}}, J_3^{\mathbb{R}} : \frac{\widetilde{h}'}{SU(2)_{\mathcal{R}}} \otimes H_H \subseteq SU(6). \tag{4.14}$$

For what concerns the remaining models, $J_{3,M}^{\mathbb{C}}$ and $J_{3,M}^{\mathbb{R}}$ respectively have $n_V = 0, 1$ and thus they do not have $\mathcal{N} = 2$ non-BPS $Z = 0$ critical points of $V_{BH,\mathcal{N}=2}$ at all.

The *stu* model has $\widetilde{h}' = SO(2)$, and thus Eqs. (4.13) and (4.14) do not hold. In such a model all goes the same way as for the previously treated $\mathcal{N} = 2$, $\frac{1}{2}$-BPS case, and consequently in *stu* model $\mathcal{N} = 2$ non-BPS $Z = 0$ critical points of $V_{BH,\mathcal{N}=2}$ are stable, *i.e.* there are no "flat" non-BPS $Z = 0$ Hessian directions at all. This can be simply understood by noticing that in such an $\mathcal{N} = 2$ framework *triality symmetry* puts non-BPS $Z = 0$ critical points on the very same footing of $\frac{1}{2}$-BPS critical points, which are always stable and thus do not have any "flat" direction at all.

	non-BPS $Z=0$ orbit $\mathcal{O}_{non-BPS,Z=0}=\frac{G_V}{\widetilde{H}}$	$\widetilde{h}'\equiv\frac{m.c.s.(\widetilde{H})}{U(1)}$	$\frac{H_H}{SU(2)_H}$
$J_3^{\mathbb{H}}$	$\frac{SO^*(12)}{SU(4,2)}$	$SU(4)\otimes SU(2)_{\mathcal{R}}$	$\nexists H_H,\quad SU_H(2)=G_H$
$J_3^{\mathbb{C}}$	$\frac{SU(3,3)}{SU(2,1)\otimes SU(1,2)}$	$SU(2)\otimes SU(2)_{\mathcal{R}}\otimes U(1)$	$U(1)$
$J_3^{\mathbb{R}}$	$\frac{Sp(6,\mathbb{R})}{SU(2,1)}$	$SU(2)_{\mathcal{R}}$	$SU(2)$
stu	$\frac{(SU(1,1))^3}{(U(1))^2}$	$SO(2)$	$(SU(2))^2\otimes SU(2)_{\mathcal{R}}$
$J_{3,M}^{\mathbb{R}}$	$-$	$-$	$USp(6)$, $SU(2)_H=SU(2)_{\mathcal{R}}$
$J_{3,M}^{\mathbb{C}}$	$-$	$-$	$SU(6)$, $SU(2)_H=SU(2)_{\mathcal{R}}$

Table 3: **The non-BPS $Z=0$ supporting BH charge orbit $\mathcal{O}_{non-BPS,Z=0}$, and the compact groups $\widetilde{h}'$ and $\frac{H_H}{SU(2)_H}$ (relevant at $\mathcal{N}=2$ non-BPS $Z=0$ critical points) for the $\mathcal{N}=2$, $d=4$ supergravities listed in Table 1**

The corresponding data for all the *maximal magic* $\mathcal{N}=2$, $d=4$ supergravities which are consistent truncations of the $\mathcal{N}=8$, $d=4$ theory (listed in Table 1) are given in Table 3 (for the column "$\widetilde{h}'$" refer to Table 8 of [21]).

Let us consider two explicit examples, namely the models $J_3^{\mathbb{H}}$ and stu.

The model $J_3^{\mathbb{H}}$ has the highest number of vector multiplets ($n_V=15$) and no hypermultiplets at all ($n_H=0$); thus, H_H cannot be defined, and $SU(2)=SU(2)_H$ is promoted to a global symmetry, which here coincides with G_H itself. $SU(2)_{\mathcal{R}}$ is identified with the factor $SU(2)$ in $\widetilde{h}'=SU(4)\otimes SU(2)$, thus it holds that $SU(4)\otimes G_H=SU(4)\otimes SU(2)_H\subset SU(6)$. The $\mathbf{15}$, $\overline{\mathbf{15}}$ and $\mathbf{20}$ of $SU(6)$ decompose under $SU(4)\otimes SU(2)_H$ as follows:

$$\begin{aligned}
\mathbf{15}&=(\mathbf{4},\mathbf{2})\oplus(\mathbf{6},\mathbf{1})\oplus(\mathbf{1},\mathbf{1});\\
\overline{\mathbf{15}}&=\left(\overline{\mathbf{4}},\mathbf{2}\right)\oplus(\mathbf{6},\mathbf{1})\oplus(\mathbf{1},\mathbf{1});\\
\mathbf{20}&=(\mathbf{4},\mathbf{1})\oplus\left(\overline{\mathbf{4}},\mathbf{1}\right)\oplus(\mathbf{6},\mathbf{2}).
\end{aligned}\tag{4.15}$$

Thus, by also recalling Eq. (3.14), one obtains that at $\mathcal{N}=2$ non-BPS $Z=0$ critical points the $\mathcal{N}=8$, $\frac{1}{8}$-BPS enhanced symmetry $SU(6)\otimes SU(2)_{\mathcal{R}}$ decomposes under $SU(4)\otimes SU(2)_H\otimes SU(2)_{\mathcal{R}}$ as follows:

$$\begin{aligned}
&m\neq 0:(\mathbf{15},\mathbf{1})\oplus\left(\overline{\mathbf{15}},\mathbf{1}\right)=(\mathbf{4},\mathbf{2},\mathbf{1})\oplus\left(\overline{\mathbf{4}},\mathbf{2},\mathbf{1}\right)\oplus(\mathbf{6},\mathbf{1},\mathbf{1})\oplus(\mathbf{6},\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{1},\mathbf{1});\\
&m=0:(\mathbf{20},\mathbf{2})=(\mathbf{4},\mathbf{1},\mathbf{2})\oplus\left(\overline{\mathbf{4}},\mathbf{1},\mathbf{2}\right)\oplus(\mathbf{6},\mathbf{2},\mathbf{2}).
\end{aligned}\tag{4.16}$$

As previously mentioned, in general the $\mathcal{N}=2$ vector multiplets' and hypermultiplets' scalar degrees of freedom are respectively given by the singlets and doublets of $SU(2)_H$. For the model under consideration, all vector multiplets' scalars are included in the $\mathcal{N}=2$, $d=4$ spectrum, whereas all hypermultiplets'

scalars are truncated away by dimensional reduction $\mathcal{N}=8 \longrightarrow \mathcal{N}=2$. Thus, the representation decomposition (4.16) yields that at $\mathcal{N}=2$ non-BPS $Z=0$ critical points the vector multiplets' scalars and hypermultiplets' scalars respectively sit in the following representations of $SU(4)\otimes SU(2)_H \otimes SU(2)_{\mathcal{R}}$:

$$
\begin{gathered}
\underset{\text{(all in the } \mathcal{N}=2,\ d=4 \text{ spectrum)}}{30 \text{ (real) vectors' scalar degrees of freedom}} = \left\{ \begin{array}{l} \overbrace{(\mathbf{6},\mathbf{1},\mathbf{1})\oplus(\mathbf{6},\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{1},\mathbf{1})}^{14 \quad m\neq 0} \oplus \\ \\ \oplus \overbrace{(\mathbf{4},\mathbf{1},\mathbf{2})\oplus(\overline{\mathbf{4}},\mathbf{1},\mathbf{2})}^{16 \quad m=0}; \end{array} \right. \\
\\
\underset{\text{(all truncated away in the } \mathcal{N}=8 \longrightarrow \mathcal{N}=2 \text{ reduction)}}{40 \text{ (real) hypers' scalar degrees of freedom}} = \overbrace{(\mathbf{4},\mathbf{2},\mathbf{1})\oplus(\overline{\mathbf{4}},\mathbf{2},\mathbf{1})}^{16 \quad m\neq 0} \oplus \overbrace{(\mathbf{6},\mathbf{2},\mathbf{2})}^{24 \quad m=0},
\end{gathered}
\tag{4.17}
$$

yielding a non-BPS $Z=0$ mass splitting "$14\ m\neq 0/16\ m=0$" of the vector multiplets' scalar degrees of freedom, matching the result obtained in [21].

The model *stu* is the one with the smallest number of vector multiplets ($n_V=3$) still exhibiting non-BPS $Z=0$ critical points. Without loss of generality (due to *triality symmetry*), one can identify $SU(2)_{\mathcal{R}}$ with the fourth factor $SU(2)$ in $H_H = SO(4)\otimes SO(4) = (SU(2))^4$, whereas the $\mathcal{N}=2$ $\mathcal{R}$-symmetry can be identified with the third factor $SU(2)$ in H_H. Thus, as yielded by Table 3, the $\mathcal{N}=2$ non-BPS $Z=0$ symmetry $\widetilde{h}'\otimes H_H$ can be rewritten as

$$
stu: \widetilde{h}'\otimes H_H = SO(2)\otimes (SU(2))^2 \otimes SU(2)_H \otimes SU(2)_{\mathcal{R}}. \tag{4.18}
$$

Thus, it holds that $SO(2)\otimes (SU(2))^2\otimes SU(2)_H \subset SU(6)$.

Thus, by also recalling Eq. (3.14), one obtains that at $\mathcal{N}=2$ non-BPS $Z=0$ critical points the $\mathcal{N}=8$, $\frac{1}{8}$-BPS enhanced symmetry $SU(6)\otimes SU(2)_{\mathcal{R}}$ decomposes under $(SU(2))^2\otimes SU(2)_H\otimes SU(2)_{\mathcal{R}}$ as follows:

$$
\begin{aligned}
& m\neq 0: (\mathbf{15},\mathbf{1})\oplus(\overline{\mathbf{15}},\mathbf{1}) = 6\,(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})\oplus 2\,(\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1})\oplus 2\,(\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1})\oplus 2\,(\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1})\,; \\
& m=0: (\mathbf{20},\mathbf{2}) = (\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2})\oplus 2\,(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2})\oplus 2\,(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2})\oplus 2\,(\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2})\,.
\end{aligned}
\tag{4.19}
$$

Such a representation decomposition yields that at $\mathcal{N}=2$ non-BPS $Z=0$ critical points the vector multiplets' scalars and hypermultiplets' scalars respectively sit in the following representations of $(SU(2))^2\otimes SU(2)_H\otimes SU(2)_{\mathcal{R}}$:

$$
\begin{gathered}
\underset{\text{(6 in the } \mathcal{N}=2,\ d=4 \text{ spectrum, 24 truncated away)}}{30 \text{ (real) vectors' scalar degrees of freedom}} = \left\{ \begin{array}{l} \underset{6 \text{ in the } \mathcal{N}=2,\ d=4 \text{ spectrum}}{\overbrace{6\,(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})}^{m\neq 0}} \oplus \\ \\ \oplus \overbrace{2\,(\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1})}^{m\neq 0} \oplus \underset{24 \text{ truncated away}}{\overbrace{2\,(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2})\oplus 2\,(\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2})}^{m=0}}; \end{array} \right. \\
\\
\underset{\text{(16 in the } \mathcal{N}=2,\ d=4 \text{ spectrum, 24 truncated away)}}{40 \text{ (real) hypers' scalar degrees of freedom}} = \left\{ \begin{array}{l} \underset{16 \text{ in the } \mathcal{N}=2,\ d=4 \text{ spectrum}}{\overbrace{(\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2})}^{m=0}} \oplus \\ \\ \oplus \overbrace{2\,(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2})}^{m=0} \oplus \underset{24 \text{ truncated away}}{\overbrace{2\,(\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1})\oplus 2\,(\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1})}^{m\neq 0}}, \end{array} \right.
\end{gathered}
\tag{4.20}
$$

yielding that the Hessian of $V_{BH,\mathcal{N}=2}$ has no "flat" directions at non-BPS $Z=0$ critical points in the *stu* model. As mentioned above, this can be traced back to the noteworthy *triality symmetry* of the model

under consideration, putting non-BPS $Z=0$ critical points on the very same footing of $\frac{1}{2}$-BPS critical points.

Thus, in this sense one can state that in the *stu* model the stability of $\frac{1}{2}$-BPS critical points implies, by *triality symmetry*, the stability of non-BPS $Z=0$ critical points. This can be quantitatively understood by considering the representation decomposition of $SU(6)\otimes SU(2)_{\mathcal{R}}$ in the $\frac{1}{2}$-BPS case. In such a case $SU(2)_{\mathcal{R}}=SU(2)_H$, and $SU(6)\otimes SU(2)_{\mathcal{R}}$ decomposes into $H_0\otimes\frac{H_H}{SU(2)_{\mathcal{R}}}\otimes SU(2)_{\mathcal{R}}=(U(1))^2\otimes(SU(2))^3\otimes SU(2)_{\mathcal{R}}$ (once again, the choice of $SU(2)_{\mathcal{R}}$ as the fourth $SU(2)$ does not imply any loss of generality, due to *triality symmetry*). It is thus easy to realize that this amounts simply to interchange the third and fourth $SU(2)$s in the representation decomposition (4.19).

5 $\mathcal{N}=8$ non-BPS Critical Points and $\mathcal{N}=2$ non-BPS $Z\neq 0$ Critical Points

For the $\mathcal{N}=2$, $d=4$ supergravities listed in Table 1, the overall symmetry $\mathcal{S}_{non-BPS,Z\neq 0}$ of $\mathcal{N}=2$, $d=4$ non-BPS $Z\neq 0$ critical points of $V_{BH,\mathcal{N}=2}$ is given by [21]

$$\mathcal{S}_{non-BPS,Z\neq 0}=\widehat{h}\otimes H_H, \tag{5.1}$$

where $\widehat{h}$ is the *m.c.s.* of the stabylizer $\widehat{H}$ of the $\mathcal{N}=2$ non-BPS $Z\neq 0$-supporting BH charge orbit [21]. Furthermore, $\mathcal{N}=2$ non-BPS $Z\neq 0$ case has $\mathcal{N}=2$ quartic G_V-invariant $I_4<0$. Thus, it is clear that $\mathcal{N}=2$ non-BPS $Z\neq 0$ case comes from the $\mathcal{N}=8\longrightarrow\mathcal{N}=2$ supersymmetry reduction given by Eq. (3.16). Thus, $\mathcal{S}_{non-BPS,Z\neq 0}$ must be included in the overall enhanced symmetry $USp(8)$ of the $\mathcal{N}=8$ non-BPS case:

$$\mathcal{S}_{non-BPS,Z\neq 0}\subsetneq USp(8). \tag{5.2}$$

It is worth pointing out that at $\mathcal{N}=2$ non-BPS $Z\neq 0$ critical points of $V_{BH,\mathcal{N}=2}$ the group $SU(2)_{\mathcal{R}}$ cannot be defined, and in general the $\mathcal{N}=2$ $\mathcal{R}$-symmetry $SU(2)_H\subsetneq H_H$, with the exception of the model $J_3^{\mathbb{H}}$, in which $n_H=0$ and thus H_H cannot be defined and $SU(2)_H=G_H$ is a global symmetry.

In order to determine the mass degeneracy pattern of the Hessian of $V_{BH,\mathcal{N}=2}$ at $\mathcal{N}=2$ non-BPS $Z\neq 0$ critical points, one will thus have to consider the decomposition of the representations $\mathbf{42}$ ($m=0$), $\mathbf{27}$ ($m\neq 0$) and $\mathbf{1}$ ($m\neq 0$) of the enhanced $\mathcal{N}=8$ non-BPS symmetry $USp(8)$ (recall Eqs. (3.16) and (3.19)) into suitable representations of $\mathcal{S}_{non-BPS,Z\neq 0}$. The embedding (5.2) is apriori not unique, but only one embedding among the possible ones is consistent with the known quantum numbers of the vector and hyper multiplets' scalars in the consider models, and thus consistent with the performed supersymmetry reduction $\mathcal{N}=8\longrightarrow\mathcal{N}=2$.

The corresponding data for all the $\mathcal{N}=2$, $d=4$ supergravities which are consistent truncations of the $\mathcal{N}=8$, $d=4$ theory (listed in Table 1) are given in Table 4 (for the column "$\widehat{h}$" refer to Table 8 of [21]).

In the following Subsects. we will analyze each model separately.

5.1 $J_3^{\mathbb{H}}$

As given by Table 1, this model has $(n_V,n_H)=(15,0)$, and $\frac{G_V}{H_V}=\frac{SO^*(12)}{U(6)}$. H_H cannot be defined, and $SU(2)_H=G_H$ is the global symmetry due to $n_H=0$. From Table 2 of [37] the fundamental representation $\mathbf{56}$ of $G=E_{7(7)}$ decomposes along $G_V\otimes G_H=SO^*(12)\otimes SU(2)_H$ as follows:

$$\mathbf{56}\longrightarrow(\mathbf{32},\mathbf{1})\oplus(\mathbf{12},\mathbf{2}), \tag{5.3}$$

yielding that the 32 real electric and magnetic charges $\left\{p^0,p^1,....,p^{15},q_0,q_1,...q_{15}\right\}$ of the $1+15$ vectors of $J_3^{\mathbb{H}}$ lie in the $SU(2)_H$-singlet real representation $(\mathbf{32},\mathbf{1})$ of $SO^*(12)\otimes SU(2)_H$ (here and in what follows the index "0" pertains to the graviphoton). On the other hand, the fundamental representation $\mathbf{8}$ of the enhanced $\mathcal{N}=8$ non-BPS symmetry $USp(8)$ decomposes along $\mathcal{S}_{non-BPS,Z\neq 0}=\widehat{h}\otimes SU(2)_H=USp(6)\otimes SU(2)_H$ as follows:

$$\mathbf{8}\longrightarrow(\mathbf{6},\mathbf{1})\oplus(\mathbf{1},\mathbf{2}). \tag{5.4}$$

	non-BPS, $Z \neq 0$ orbit $\mathcal{O}_{non-BPS,Z\neq 0} = \frac{G_V}{\widehat{H}}$	$\widehat{h} \equiv m.c.s.\left(\widehat{H}\right)$	$\frac{H_H}{SU(2)_H}$
$J_3^{\mathbb{H}}$	$\frac{SO^*(12)}{SU^*(6)}$	$USp(6)$	$\nexists H_H, \quad SU_H(2) = G_H$
$J_3^{\mathbb{C}}$	$\frac{SU(3,3)}{SL(3,\mathbb{C})}$	$SU(3)$	$U(1)$
$J_3^{\mathbb{R}}$	$\frac{Sp(6,\mathbb{R})}{SL(3,\mathbb{R})}$	$SU(2)$	$SU(2)$
stu	$\frac{(SU(1,1))^3}{(SO(1,1))^2}$	$\mathbb{I}$	$(SU(2))^3$
$J_{3,M}^{\mathbb{R}}$	$SU(1,1)$	$\mathbb{I}$	$USp(6)$
$J_{3,M}^{\mathbb{C}}$	$-$	$-$	$SU(6)$

Table 4: **The non-BPS $Z \neq 0$ supporting BH charge orbit $\mathcal{O}_{non-BPS,Z\neq 0}$, and the compact groups $\widehat{h}$ and $\frac{H_H}{SU(2)_H}$ (relevant at $\mathcal{N}=2$ non-BPS $Z \neq 0$ critical points) for the $\mathcal{N}=2$, $d=4$ supergravities listed in Table 1**

The decomposition of the representations **42**, **27** and **1** of $USp(8)$ along $\mathcal{S}_{non-BPS,Z\neq 0}$ and its interpretation in terms of the $\mathcal{N}=2$, $d=4$ spectrum (and of the truncated scalar degrees of freedom) reads as follows:

$$m=0: \mathbf{42} \longrightarrow \left\{ \begin{array}{l} \overbrace{(\mathbf{14'},\mathbf{2})}^{28\ m=0\ \text{hypers' scalar degrees of freedom truncated away}} \oplus \\ \oplus \overbrace{(\mathbf{14},\mathbf{1})}^{14\ m=0\ \text{vectors' scalar degrees of freedom in}\ \mathcal{N}=2,\ d=4\ \text{spectrum}} ; \end{array} \right.$$

$$m \neq 0: \left\{ \begin{array}{l} \mathbf{27} \longrightarrow \left\{ \begin{array}{l} \overbrace{(\mathbf{6},\mathbf{2})}^{12\ m\neq 0\ \text{hypers' scalar degrees of freedom truncated away}} \oplus \\ \oplus \overbrace{(\mathbf{1},\mathbf{1}) \oplus (\mathbf{14},\mathbf{1})}^{15\ m\neq 0\ \text{vectors' scalar degrees of freedom in}\ \mathcal{N}=2,\ d=4\ \text{spectrum}} ; \end{array} \right. \\ \\ \mathbf{1} \longrightarrow \overbrace{(\mathbf{1},\mathbf{1})}^{1\ m\neq 0\ \text{vectors' scalar degree of freedom in}\ \mathcal{N}=2,\ d=4\ \text{spectrum}} , \end{array} \right. \tag{5.5}$$

where **14** and **14′** respectively stand for the two-fold and three-fold antisymmetric (traceless) of $USp(6)$.

It should be noticed that for $J_3^{\mathbb{H}}$ the embedding of $\mathcal{S}_{non-BPS,Z\neq 0}$ in the enhanced $\mathcal{N}=8$ non-BPS symmetry $USp(8)$ is unique. Moreover, since $J_3^{\mathbb{H}}$ has the highest number $n_V = 15$ of Abelian vector

multiplets, all (would-be $\mathcal{N}=2$) vectors' scalar degrees of freedom of the starting $\mathcal{N}=8$ theory survive after the reduction $\mathcal{N}=8 \longrightarrow \mathcal{N}=2$.

The $\mathcal{N}=2$ non-BPS $Z \neq 0$ mass degeneracy pattern of the vector multiplets' scalar degrees of freedom resulting from the decomposition (5.5) is "$n_V+1=16$ $m\neq 0$ / $n_V-1=14$ $m=0$", thus confirming the Hessian splitting found in [10].

5.2 $J_3^{\mathbb{C}}$

As given by Table 1, this model has $(n_V, n_H) = (9,1)$, and $\frac{G_V}{H_V} \otimes \frac{G_H}{H_H} = \frac{SU(3,3)}{SU(3)\otimes SU(3)\otimes U(1)} \otimes \frac{SU(2,1)}{SU(2)_H \otimes U(1)}$. From Table 2 of [37] the fundamental representation **56** of $G = E_{7(7)}$ decomposes along $G_V \otimes G_H = SU(3,3) \otimes SU(2,1)$ as follows:

$$\mathbf{56} \longrightarrow (\mathbf{20},\mathbf{1}) \oplus (\mathbf{6},\mathbf{3}) \oplus \left(\overline{\mathbf{6}},\overline{\mathbf{3}}\right), \tag{5.6}$$

yielding that the 20 real electric and magnetic charges $\left\{p^0, p^1, ..., p^9, q_0, q_1, ...q_9\right\}$ of the $1+9$ vectors of $J_3^{\mathbb{C}}$ lie in the $SU(2,1)$-singlet real representation $(\mathbf{20},\mathbf{1})$ of $SU(3,3) \otimes SU(2,1)$. On the other hand, the fundamental representation **8** of the enhanced $\mathcal{N}=8$ non-BPS symmetry $USp(8)$ decomposes along $\mathcal{S}_{non-BPS,Z\neq 0} = \widehat{h} \otimes H_H = SU(3) \otimes SU(2)_H \otimes U(1)$ as follows (here and in what follows we disregard the quantum numbers of $U(1)$, not essential for our purposes):

$$\mathbf{8} \longrightarrow (\mathbf{3},\mathbf{1}) \oplus \left(\overline{\mathbf{3}},\mathbf{1}\right) \oplus (\mathbf{1},\mathbf{2}). \tag{5.7}$$

The decomposition of the representations **42**, **27** and **1** of $USp(8)$ along $\mathcal{S}_{non-BPS,Z\neq 0}$ and its interpretation in terms of the $\mathcal{N}=2$, $d=4$ spectrum (and of the truncated scalar degrees of freedom) reads as follows:

$$m=0: \mathbf{42} \longrightarrow \left\{ \begin{array}{l} \overbrace{(\mathbf{1},\mathbf{2}) \oplus (\mathbf{1},\mathbf{2})}^{4\ m=0 \text{ hypers' scalar degrees of freedom in } \mathcal{N}=2,\ d=4 \text{ spectrum}} \oplus \\ \oplus \overbrace{\left(\overline{\mathbf{3}},\mathbf{1}\right) \oplus (\mathbf{3},\mathbf{1})}^{6\ m=0 \text{ vectors' scalar degrees of freedom truncated away}} \oplus \\ \oplus \overbrace{\left(\overline{\mathbf{6}},\mathbf{2}\right) \oplus (\mathbf{6},\mathbf{2})}^{24\ m=0 \text{ hypers' scalar degrees of freedom truncated away}} \oplus \\ \oplus \overbrace{(\mathbf{8},\mathbf{1})}^{8\ m=0 \text{ vectors' scalar degrees of freedom in } \mathcal{N}=2,\ d=4 \text{ spectrum}} ; \end{array} \right.$$

$$m\neq 0: \left\{ \begin{array}{l} \mathbf{27} \longrightarrow \left\{ \begin{array}{l} \overbrace{\left(\overline{\mathbf{3}},\mathbf{1}\right) \oplus (\mathbf{3},\mathbf{1})}^{6\ m\neq 0 \text{ vectors' scalar degrees of freedom truncated away}} \oplus \\ \oplus \overbrace{(\mathbf{8},\mathbf{1})}^{8\ m\neq 0 \text{ vectors' scalar degrees of freedom in } \mathcal{N}=2,\ d=4 \text{ spectrum}} \oplus \\ \overbrace{\left(\overline{\mathbf{3}},\mathbf{2}\right) \oplus (\mathbf{3},\mathbf{2})}^{12\ m\neq 0 \text{ hypers' scalar degrees of freedom truncated away}} \oplus \\ \oplus \overbrace{(\mathbf{1},\mathbf{1})}^{1\ m\neq 0 \text{ vectors' scalar degree of freedom in } \mathcal{N}=2,\ d=4 \text{ spectrum}} ; \end{array} \right. \\ \\ \mathbf{1} \longrightarrow \overbrace{(\mathbf{1},\mathbf{1})}^{1\ m\neq 0 \text{ vectors' scalar degree of freedom in } \mathcal{N}=2,\ d=4 \text{ spectrum}} . \end{array} \right. \tag{5.8}$$

It should be noticed that for $J_3^{\mathbb{C}}$ the embedding of $\mathcal{S}_{non-BPS,Z\neq 0}$ in the enhanced $\mathcal{N}=8$ non-BPS symmetry $USp(8)$ is apriori not unique, but the only consistent with the $\mathcal{N}=8 \longrightarrow \mathcal{N}=2$ reduction originating $J_3^{\mathbb{C}}$ is the following two-step one:

$$USp(8) \supsetneq USp(6) \otimes USp(2) \supsetneq SU(3) \otimes SU(2)_H \otimes U(1). \tag{5.9}$$

Moreover, as evident from the decomposition (5.8), the $\mathcal{N}=8 \longrightarrow \mathcal{N}=2$ reduction originating $J_3^{\mathbb{C}}$ truncates away:

1) 6 $m=0$ and 6 $m\neq 0$ vectors' scalar degrees of freedom, both sets sitting in the $\left(\overline{\mathbf{3}},\mathbf{1}\right)\oplus(\mathbf{3},\mathbf{1})$ of $SU(3)\otimes SU(2)_H$;

2) 24 $m=0$ and 12 $m\neq 0$ hypers' scalar degrees of freedom, respectively sitting in the $\left(\overline{\mathbf{6}},\mathbf{2}\right)\oplus(\mathbf{6},\mathbf{2})$ and $\left(\overline{\mathbf{3}},\mathbf{2}\right)\oplus(\mathbf{3},\mathbf{2})$ of $SU(3)\otimes SU(2)_H$.

The resulting $\mathcal{N}=2$ $J_3^{\mathbb{C}}$ spectrum is composed by 4 $m=0$ real hypers' scalar degrees of freedom (rearranging in 1 quaternionic hypermultiplet scalar), and by $n_V+1=10$ $m\neq 0$ and $n_V-1=8$ $m=0$ real vectors' scalar degrees of freedom, whose mass degeneracy pattern thus confirms the Hessian splitting found in [10].

5.3 $J_3^{\mathbb{R}}$

As given by Table 1, this model has $(n_V,n_H)=(6,2)$, and $\frac{G_V}{H_V}\otimes\frac{G_H}{H_H}=\frac{Sp(6,\mathbb{R})}{U(3)}\otimes\frac{G_{2(2)}}{SU(2)\otimes SU(2)_H}$. From Table 2 of [37] the fundamental representation $\mathbf{56}$ of $G=E_{7(7)}$ decomposes along $G_V\otimes G_H=Sp(6,\mathbb{R})\otimes G_{2(2)}$ as follows:

$$\mathbf{56}\longrightarrow(\mathbf{14'},\mathbf{1})\oplus(\mathbf{6},\mathbf{7}), \tag{5.10}$$

where $\mathbf{14'}$ is the three-fold antisymmetric (traceless) representation of $Sp(6,\mathbb{R})$. The decomposition (5.10) yields that the 14 real electric and magnetic charges $\left\{p^0,p^1,....,p^6,q_0,q_1,...q_6\right\}$ of the $1+6$ vectors of $J_3^{\mathbb{R}}$ lie in the $G_{2(2)}$-singlet real representation $(\mathbf{14'},\mathbf{1})$ of $Sp(6,\mathbb{R})\otimes G_{2(2)}$. The symmetry group $\mathcal{S}_{non-BPS,Z\neq 0}$ of $J_3^{\mathbb{R}}$ reads

$$\mathcal{S}_{non-BPS,Z\neq 0}=\widehat{h}\otimes H_H=SO(3)\otimes SU(2)\otimes SU(2). \tag{5.11}$$

Thus, apriori $\mathcal{S}_{non-BPS,Z\neq 0}$ can be embedded in the enhanced $\mathcal{N}=8$ non-BPS symmetry $USp(8)$ in many ways, but the only consistent with the $\mathcal{N}=8\longrightarrow\mathcal{N}=2$ reduction originating $J_3^{\mathbb{R}}$ is the following two-step one:

$$\begin{array}{ccccc} USp(8)\supsetneq & USp(4) & \otimes & USp(4) & ,\quad H_H=SU(2)_P\otimes SU(2)_H, \\ & \cup & & \cup & \\ & SU(2)_P & \otimes & SU(2)\otimes SU(2)_H, & \end{array} \tag{5.12}$$

yielding that $\mathcal{S}_{non-BPS,Z\neq 0}$ can be rewritten as

$$\mathcal{S}_{non-BPS,Z\neq 0}=SU(2)_P\otimes SU(2)\otimes SU(2)_H, \tag{5.13}$$

where $SU(2)_P=\frac{H_H}{SU(2)_H}$ is the $SU(2)$-*principal embedding*[6] of one (say, without any loss of generality, of the first) of the two $USp(4)$, thus sitting in a spin $s=\frac{3}{2}$ representation $(\mathbf{4},\mathbf{1},\mathbf{1})$ with respect to $SU(2)_P\otimes SU(2)\otimes SU(2)_H$. The identification $H_H=SU(2)_P\otimes SU(2)_H$ is consistent with the known result that the hypermultiplets' quaternionic scalars of $J_3^{\mathbb{R}}$ have spins $(s,s')=\left(\frac{3}{2},\frac{1}{2}\right)$ with respect to H_H, and thus sit in a representation $(\mathbf{4},\mathbf{2})$ of such a stabylizer, where the spin $s'=\frac{1}{2}$ is with respect to the $\mathcal{N}=2$ $\mathcal{R}$-symmetry $SU(2)_H$ in H_H. Thus, the fundamental representation $\mathbf{8}$ of the enhanced $\mathcal{N}=8$ non-BPS symmetry $USp(8)$ decomposes along $SU(2)_P\otimes SU(2)\otimes SU(2)_H$ as follows:

$$\mathbf{8}\longrightarrow(\mathbf{1},\mathbf{2},\mathbf{1})\oplus(\mathbf{4},\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{1},\mathbf{2}). \tag{5.14}$$

The decomposition of the representations $\mathbf{42}$, $\mathbf{27}$ and $\mathbf{1}$ of $USp(8)$ along $\mathcal{S}_{non-BPS,Z\neq 0}$ and its interpretation in terms of the $\mathcal{N}=2$, $d=4$ spectrum (and of the truncated scalar degrees of freedom)

[6] The group sequence $USp(n)_{n\in\mathbb{N}}$ has an embedding, called *principal*, in $SU(2)$ with spin $s_n=n-\frac{1}{2}$ [63].

reads as follows:

$$
m=0:\mathbf{42}\longrightarrow\left\{\begin{array}{lcr}
 & \overbrace{(\mathbf{5},\mathbf{1},\mathbf{1})}^{\text{5 } m\text{=0 vectors' scalar degrees of freedom in } \mathcal{N}\text{=2, } d\text{=4 spectrum}} & \oplus \\
\oplus & \overbrace{(\mathbf{5},\mathbf{2},\mathbf{2})}^{\text{20 } m\text{=0 hypers' scalar degrees of freedom truncated away}} & \oplus \\
\oplus & \overbrace{(\mathbf{4},\mathbf{1},\mathbf{2})}^{\text{8 } m\text{=0 hypers' scalar degrees of freedom in } \mathcal{N}\text{=2, } d\text{=4 spectrum}} & \oplus \\
\oplus & \overbrace{(\mathbf{4},\mathbf{2},\mathbf{1})\oplus(\mathbf{1},\mathbf{1},\mathbf{1})}^{\text{9 } m\text{=0 vectors' scalar degrees of freedom truncated away}} & ;
\end{array}\right.
$$

$$
m\neq 0:\left\{\begin{array}{l}
\mathbf{27}\longrightarrow\left\{\begin{array}{lcr}
 & \overbrace{(\mathbf{5},\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{1},\mathbf{1})}^{\text{6 } m\neq\text{0 vectors' scalar degrees of freedom in } \mathcal{N}\text{=2, } d\text{=4 spectrum}} & \oplus \\
\oplus & \overbrace{(\mathbf{4},\mathbf{2},\mathbf{1})\oplus(\mathbf{1},\mathbf{1},\mathbf{1})}^{\text{9 } m\text{=0 vectors' scalar degrees of freedom truncated away}} & \oplus \\
 & \overbrace{(\mathbf{4},\mathbf{1},\mathbf{2})\oplus(\mathbf{1},\mathbf{2},\mathbf{2})}^{\text{12 } m\neq\text{0 hypers' scalar degrees of freedom truncated away}} & ;
\end{array}\right. \\
\\
\mathbf{1}\longrightarrow\quad\overbrace{(\mathbf{1},\mathbf{1},\mathbf{1})}^{\text{1 } m\neq\text{0 vectors' scalar degree of freedom in } \mathcal{N}\text{=2, } d\text{=4 spectrum}}\quad .
\end{array}\right.
\tag{5.15}
$$

Such decompositions yield that the $\mathcal{N}=8\longrightarrow\mathcal{N}=2$ reduction originating $J_3^{\mathbb{R}}$ truncates away:

1) 9 $m=0$ and 9 $m\neq 0$ vectors' scalar degrees of freedom, both sets sitting in the $(\mathbf{4},\mathbf{2},\mathbf{1})\oplus(\mathbf{1},\mathbf{1},\mathbf{1})$ of $SU(2)_P\otimes SU(2)\otimes SU(2)_H$;

2) 20 $m=0$ and 12 $m\neq 0$ hypers' scalar degrees of freedom, respectively sitting in the $(\mathbf{5},\mathbf{2},\mathbf{2})$ and $(\mathbf{4},\mathbf{1},\mathbf{2})\oplus(\mathbf{1},\mathbf{2},\mathbf{2})$ of $SU(2)_P\otimes SU(2)\otimes SU(2)_H$.

The resulting $\mathcal{N}=2$ $J_3^{\mathbb{R}}$ spectrum is composed by 8 $m=0$ real hypers' scalar degrees of freedom (rearranging in 2 quaternionic hypermultiplet scalar), and by $n_V+1=7$ $m\neq 0$ and $n_V-1=5$ $m=0$ real vectors' scalar degrees of freedom, whose mass degeneracy pattern thus confirms once again the Hessian splitting found in [10].

5.4 *stu*

As given by Table 1, this model has $(n_V,n_H)=(3,4)$, and $\frac{G_V}{H_V}\otimes\frac{G_H}{H_H}=\frac{SU(1,1)}{U(1)}\otimes\frac{SO(2,2)}{SO(2)\otimes SO(2)}\otimes\frac{SO(4,4)}{SO(4)\otimes SO(4)}$. From Eq. (182) of [35] the fundamental representation $\mathbf{56}$ of $G=E_{7(7)}$ decomposes along $G_V\otimes G_H=SU(1,1)\otimes SO(2,2)\otimes SO(4,4)\sim(SU(1,1))^3\otimes SO(4,4)$ as follows (the three $SU(1,1)$ are actually indistinguishable due to *triality symmetry*):

$$
\mathbf{56}\longrightarrow(\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{1})\oplus(\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{8}_v)\oplus(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{8}_s)\oplus(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{8}_c)\,, \tag{5.16}
$$

where $\mathbf{8}_v$, $\mathbf{8}_s$ and $\mathbf{8}_c$ respectively are the vector, chiral spinorial and anti-chiral spinorial representations of $SO(4,4)$. The decomposition (5.16) yields that the 8 real electric and magnetic charges $\left\{p^0,p^1,....,p^3,q_0,q_1,...q_3\right\}$ of the $1+3$ vectors of the *stu* model lie in the $SO(4,4)$-singlet real representation $(\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{1})$ of $(SU(1,1))^3\otimes SO(4,4)$. The symmetry group $\mathcal{S}_{non-BPS,Z\neq 0}$ of the *stu* model

reads

$$\mathcal{S}_{non-BPS,Z\neq 0}=\widehat{h}\otimes H_H \overset{\widehat{h}_{stu}=\mathbb{I}}{=} H_H = SO(4)\otimes SO(4)\sim (SU(2))^4 . \tag{5.17}$$

Thus, apriori $\mathcal{S}_{non-BPS,Z\neq 0}$ can be embedded in the enhanced $\mathcal{N}=8$ non-BPS symmetry $USp(8)$ in many ways, but the only consistent with the $\mathcal{N}=8\longrightarrow\mathcal{N}=2$ reduction originating the stu model is the following two-step one:

$$USp(8)\supsetneq USp(4)\otimes USp(4)\supsetneq SO(4)\otimes SO(4)\sim (SU(2))^4 . \tag{5.18}$$

We can choose the $\mathcal{N}=2$ $\mathcal{R}$-symmetry $SU(2)_H$ to be the fourth one in $\mathcal{S}_{non-BPS,Z\neq 0}$ (as we will see below, such an arbitrariness in the choice of the placement of the $\mathcal{N}=2$ $\mathcal{R}$-symmetry inside H_H is actually removed by the *triality symmetry* of the stu model). Consequently, $\mathcal{S}_{non-BPS,Z\neq 0}$ can be rewritten as

$$\mathcal{S}_{non-BPS,Z\neq 0}=(SU(2))^3\otimes SU(2)_H . \tag{5.19}$$

Thus, the fundamental representation $\mathbf{8}$ of the enhanced $\mathcal{N}=8$ non-BPS symmetry $USp(8)$ decomposes along the chain of branchings (5.18) as follows:

$$\underset{USp(8)}{\mathbf{8}}\longrightarrow \underset{USp(4)\otimes USp(4)}{(\mathbf{4},\mathbf{1})\oplus(\mathbf{1},\mathbf{4})}\longrightarrow \underset{SO(4)\otimes SO(4)}{(\mathbf{4}_s,\mathbf{1})\oplus(\mathbf{1},\mathbf{4}_s)}\longrightarrow \underset{SU(2)\otimes SU(2)\otimes SU(2)\otimes SU(2)_H}{(\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1})\oplus(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})}, \tag{5.20}$$

where $\mathbf{4}_s$ is the spinorial of $SO(4)$ (or, equivalently, the reduction of the fundamental of $USp(4)$ with respect to $SO(4)$).

Due to the chain of group inclusions (5.18) needed in the stu model in order to correctly embed $\mathcal{S}_{non-BPS,Z\neq 0}$ into $USp(8)$, the decomposition of the representations $\mathbf{42}$, $\mathbf{27}$ and $\mathbf{1}$ of $USp(8)$ along $\mathcal{S}_{non-BPS,Z\neq 0}$ should better be performed in two steps:

i) decomposition of $USp(8)$ along $USp(4)\otimes USp(4)$. It respectively yields (the prime distinguishes the - representations of the - two $USp(4)$)

$$m=0:\mathbf{42}\longrightarrow\left(\mathbf{4},\mathbf{4}'\right)\oplus\left(\mathbf{5},\mathbf{5}'\right)\oplus\left(\mathbf{1},\mathbf{1}'\right);$$

$$m\neq 0:\left\{\begin{array}{l}\mathbf{27}\longrightarrow\left(\mathbf{4},\mathbf{4}'\right)\oplus\left(\mathbf{5},\mathbf{1}'\right)\oplus\left(\mathbf{1},\mathbf{5}'\right)\oplus\left(\mathbf{1},\mathbf{1}'\right);\\ \\ \mathbf{1}\longrightarrow\left(\mathbf{1},\mathbf{1}'\right).\end{array}\right. \tag{5.21}$$

ii) Decomposition of $USp(4)\otimes USp(4)$ along $SO(4)\otimes SO(4)$. It will involve the representations $\mathbf{4}_s$ (previously introduced) and $\mathbf{4}_v$ (vector representation of $SO(4)$ or, equivalently, reduction of the antisymmetric traceless of $USp(4)$ with respect to $SO(4)$). By exploiting the following decompositions of the representations $\left(\mathbf{4},\mathbf{4}'\right)$, $\left(\mathbf{5},\mathbf{5}'\right)$, $\left(\mathbf{5},\mathbf{1}'\right)$ and $\left(\mathbf{1},\mathbf{1}'\right)$ of $USp(4)\otimes USp(4)$ along $SO(4)\otimes SO(4)$:

$$\begin{array}{l}\left(\mathbf{4},\mathbf{4}'\right)\longrightarrow\left(\mathbf{4}_s,\mathbf{4}_s'\right);\\ \\ \left(\mathbf{5},\mathbf{5}'\right)\longrightarrow\left(\mathbf{4}_v,\mathbf{4}_v'\right)\oplus\left(\mathbf{4}_v,\mathbf{1}'\right)\oplus\left(\mathbf{1},\mathbf{4}_v'\right)\oplus\left(\mathbf{1},\mathbf{1}'\right);\\ \\ \left(\mathbf{5},\mathbf{1}'\right)\longrightarrow\left(\mathbf{4}_v,\mathbf{1}'\right)\oplus\left(\mathbf{1},\mathbf{1}'\right);\\ \\ \left(\mathbf{1},\mathbf{1}'\right)\longrightarrow\left(\mathbf{1},\mathbf{1}'\right),\end{array} \tag{5.22}$$

one gets the following decompositions of representations **42**, **27** and **1** of $USp(8)$ along $SO(4) \otimes SO(4)$:

$$
\begin{aligned}
&m = 0 : \mathbf{42} \longrightarrow \left(\mathbf{4}_s, \mathbf{4}'_s\right) \oplus \left(\mathbf{4}_v, \mathbf{4}'_v\right) \oplus \left(\mathbf{4}_v, \mathbf{1}'\right) \oplus \left(\mathbf{1}, \mathbf{4}'_v\right) \oplus 2\left(\mathbf{1}, \mathbf{1}'\right);\\
&m \neq 0 : \begin{cases} \mathbf{27} \longrightarrow \left(\mathbf{4}_s, \mathbf{4}'_s\right) \oplus \left(\mathbf{4}_v, \mathbf{1}'\right) \oplus \left(\mathbf{1}, \mathbf{4}'_v\right) \oplus 3\left(\mathbf{1}, \mathbf{1}'\right); \\ \mathbf{1} \longrightarrow \left(\mathbf{1}, \mathbf{1}'\right). \end{cases}
\end{aligned}
\tag{5.23}
$$

iii) Further decomposition, performed by exploiting the group isomorphism $SO(4) \sim SU(2) \otimes SU(2)$. Under the group isomorphism $SO(4) \sim (SU(2))^2$, $\mathbf{4}_s$ and $\mathbf{4}_v$ respectively decompose as follows:

$$
\begin{aligned}
\mathbf{4}_s &\longrightarrow (\mathbf{2}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2});\\
\mathbf{4}_v &\longrightarrow (\mathbf{2}, \mathbf{2}).
\end{aligned}
\tag{5.24}
$$

Thus, the decomposition of representations **42**, **27** and **1** of $USp(8)$ along $(SU(2))^4 = (SU(2))^3 \otimes SU(2)_H$ (embedded into $USp(8)$ in the way given by the chain (5.18) of group inclusions), and its interpretation in terms of the $\mathcal{N} = 2$, $d = 4$ spectrum (and of the truncated scalar degrees of freedom), reads as follows:

$$
m = 0 : \mathbf{42} \longrightarrow \begin{cases}
\overbrace{(\mathbf{2}, \mathbf{1}, \mathbf{2}', \mathbf{1}') \oplus (\mathbf{1}, \mathbf{2}, \mathbf{2}', \mathbf{1}') \oplus (\mathbf{2}, \mathbf{2}, \mathbf{1}', \mathbf{1}')}^{12\ m=0 \text{ vectors' scalar degrees of freedom truncated away}} \oplus \\
\oplus \overbrace{(\mathbf{1}, \mathbf{2}, \mathbf{1}', \mathbf{2}') \oplus (\mathbf{2}, \mathbf{1}, \mathbf{1}', \mathbf{2}') \oplus (\mathbf{1}, \mathbf{1}, \mathbf{2}', \mathbf{2}')}^{12\ m=0 \text{ hypers' scalar degrees of freedom truncated away}} \oplus \\
\oplus \overbrace{(\mathbf{2}, \mathbf{2}, \mathbf{2}', \mathbf{2}')}^{16\ m=0 \text{ hypers' scalar degrees of freedom in } \mathcal{N}=2,\ d=4 \text{ spectrum}} \oplus \\
\oplus \overbrace{2\left(\mathbf{1}, \mathbf{1}, \mathbf{1}', \mathbf{1}'\right)}^{2\ m=0 \text{ vectors' scalar degrees of freedom in } \mathcal{N}=2,\ d=4 \text{ spectrum}} ;
\end{cases}
$$

$$
m \neq 0 : \begin{cases}
\mathbf{27} \longrightarrow \begin{cases}
\overbrace{(\mathbf{2}, \mathbf{1}, \mathbf{2}', \mathbf{1}') \oplus (\mathbf{1}, \mathbf{2}, \mathbf{2}', \mathbf{1}') \oplus (\mathbf{2}, \mathbf{2}, \mathbf{1}', \mathbf{1}')}^{12\ m\neq0 \text{ vectors' scalar degrees of freedom truncated away}} \oplus \\
\oplus \overbrace{(\mathbf{1}, \mathbf{2}, \mathbf{1}', \mathbf{2}') \oplus (\mathbf{2}, \mathbf{1}, \mathbf{1}', \mathbf{2}') \oplus (\mathbf{1}, \mathbf{1}, \mathbf{2}', \mathbf{2}')}^{12\ m\neq0 \text{ hypers' scalar degrees of freedom truncated away}} \oplus \\
\oplus \overbrace{3\left(\mathbf{1}, \mathbf{1}, \mathbf{1}', \mathbf{1}'\right)}^{3\ m\neq0 \text{ vectors' scalar degrees of freedom in } \mathcal{N}=2,\ d=4 \text{ spectrum}} ;
\end{cases} \\
\mathbf{1} \longrightarrow \overbrace{\left(\mathbf{1}, \mathbf{1}, \mathbf{1}', \mathbf{1}'\right)}^{1\ m\neq0 \text{ vectors' scalar degree of freedom in } \mathcal{N}=2,\ d=4 \text{ spectrum}} .
\end{cases}
\tag{5.25}
$$

Such decompositions yield that the $\mathcal{N}=8 \longrightarrow \mathcal{N}=2$ reduction originating the *stu* model truncates away:

1) 12 $m=0$ and 12 $m\neq 0$ vectors' scalar degrees of freedom, both sets sitting in the $(\mathbf{2},\mathbf{1},\mathbf{2}',\mathbf{1}')\oplus(\mathbf{1},\mathbf{2},\mathbf{2}',\mathbf{1}')\oplus(\mathbf{2},\mathbf{2},\mathbf{1}',\mathbf{1}')$ of $(SU(2))^3\otimes SU(2)_H$ (note the *triality symmetry* acting on the first three quantum numbers);

2) 12 $m=0$ and 12 $m\neq 0$ hypers' scalar degrees of freedom, both sets sitting in the $(\mathbf{1},\mathbf{2},\mathbf{1}',\mathbf{2}')\oplus(\mathbf{2},\mathbf{1},\mathbf{1}',\mathbf{2}')\oplus(\mathbf{1},\mathbf{1},\mathbf{2}',\mathbf{2}')$ of $(SU(2))^3\otimes SU(2)_H$ (note the *triality symmetry* acting on the first three quantum numbers).

As it is seen,both the vectors' and hypers' scalar degrees of freedom truncated out receive half of the contribution from the $\mathbf{42}$ (massless) of $USp(8)$ and the other half of the contribution from the $\mathbf{27}$ (massive) of $USp(8)$. As it holds in general, the massive singlet representation $\mathbf{1}$ of $USp(8)$ always appears in the $\mathcal{N}=2$, $d=4$ resulting spectrum.

The spectrum of the $\mathcal{N}=2$, $d=4$ *stu* model determined by the decompositions (5.25) is composed by 16 $m=0$ real hypers' scalar degrees of freedom (rearranging in 4 quaternionic hypermultiplet scalar), and by $n_V+1=4$ $m\neq 0$ and $n_V-1=2$ $m=0$ real vectors' scalar degrees of freedom, whose mass degeneracy pattern thus confirms once again the Hessian splitting found in [10].

5.5 $J_{3,M}^{\mathbb{R}}$

As given by Table 1, this model has $(n_V,n_H)=(1,7)$, and $\frac{G_V}{H_V}\otimes\frac{G_H}{H_H}=\frac{SU(1,1)}{U(1)}\otimes\frac{F_{4(4)}}{USp(6)\otimes SU(2)_H}$ (recall that $USp(2)\sim SU(2)$). From Table 2 of [37] the fundamental representation $\mathbf{56}$ of $G=E_{7(7)}$ decomposes along $G_V\otimes G_H=SU(1,1)\otimes F_{4(4)}$ as follows:

$$\mathbf{56}\longrightarrow(\mathbf{4},\mathbf{1})\oplus(\mathbf{2},\mathbf{26}). \tag{5.26}$$

Such a decomposition yields that the 4 real electric and magnetic charges $\left\{p^0,p^1,q_0,q_1\right\}$ of the $1+1$ vectors of $J_{3,M}^{\mathbb{R}}$ lie in the $F_{4(4)}$-singlet real representation $(\mathbf{4},\mathbf{1})$ of $SU(1,1)\otimes F_{4(4)}$. The representation $\mathbf{4}$ of $SU(1,1)$ corresponds to spin $s=\frac{3}{2}$, and this identifies $\frac{G_V}{H_V}=\frac{SU(1,1)}{U(1)}$ as a special Kähler manifold ($dim_{\mathbb{C}}=1$) with cubic holomorphic prepotential reading[7] (in a suitable system of special projective coordinates) $\mathcal{F}(t)=\lambda t^3$, $\lambda\in\mathbb{C}_0$. The symmetry group $\mathcal{S}_{non-BPS,Z\neq 0}$ of $J_{3,M}^{\mathbb{R}}$ is the same of the one of $J_3^{\mathbb{H}}$, and it reads ($\widehat{h}=\mathbb{I}$, as in the *stu* model)

$$\mathcal{S}_{non-BPS,Z\neq 0}=\widehat{h}\otimes H_H=H_H=USp(6)\otimes SU(2)_H. \tag{5.27}$$

As it holds also for $J_3^{\mathbb{H}}$, in the model $J_{3,M}^{\mathbb{R}}$ the embedding of $\mathcal{S}_{non-BPS,Z\neq 0}$ in the enhanced $\mathcal{N}=8$ non-BPS symmetry $USp(8)$ is unique. The fundamental representation $\mathbf{8}$ of $USp(8)$ decomposes along $USp(6)\otimes SU(2)_H$ as follows:

$$\mathbf{8}\longrightarrow(\mathbf{6},\mathbf{1})\oplus(\mathbf{1},\mathbf{2}). \tag{5.28}$$

The decomposition of the representations $\mathbf{42}$, $\mathbf{27}$ and $\mathbf{1}$ of $USp(8)$ along $\mathcal{S}_{non-BPS,Z\neq 0}$ and its interpretation in terms of the $\mathcal{N}=2$, $d=4$ spectrum (and of the truncated scalar degrees of freedom)

[7] For a discussion of (the $\mathcal{N}=2$, $d=4$ attractor Eqs. in the special Kähler geometry of) $\frac{SU(1,1)}{U(1)}$ with cubic holomorphic prepotential, see *e.g.* [21, 29] (and Refs. therein) and [31].

reads as follows:

$$
\begin{array}{l}
m=0:\mathbf{42}\longrightarrow\left\{
\begin{array}{l}
\overbrace{(\mathbf{14},\mathbf{1})}^{14\ m=0\ \text{vectors' scalar degrees of freedom truncated away}} \oplus \\[2ex]
\oplus\ \overbrace{(\mathbf{14}',\mathbf{2})}^{28\ m=0\ \text{hypers' scalar degrees of freedom in}\ \mathcal{N}=2,\ d=4\ \text{spectrum}} ;
\end{array}\right.
\\[6ex]
m\neq 0:\left\{
\begin{array}{l}
\mathbf{27}\longrightarrow\left\{
\begin{array}{l}
\overbrace{(\mathbf{6},\mathbf{2})}^{12\ m\neq0\ \text{hypers' scalar degrees of freedom truncated away}} \oplus \\[2ex]
\oplus\ \overbrace{(\mathbf{14},\mathbf{1})}^{14\ m\neq0\ \text{vectors' scalar degrees of freedom truncated away}} \oplus \\[2ex]
\overbrace{(\mathbf{1},\mathbf{1})}^{1\ m\neq0\ \text{vectors' scalar degree of freedom in}\ \mathcal{N}=2,\ d=4\ \text{spectrum}} ;
\end{array}\right.
\\[10ex]
\mathbf{1}\longrightarrow \overbrace{(\mathbf{1},\mathbf{1})}^{1\ m\neq0\ \text{vectors' scalar degree of freedom in}\ \mathcal{N}=2,\ d=4\ \text{spectrum}} ,
\end{array}\right.
\end{array}
\tag{5.29}
$$

where $\mathbf{14}$ and $\mathbf{14'}$ respectively stand for the two-fold and three-fold antisymmetric (traceless) of $USp(6)$.

Such decompositions yield that the $\mathcal{N}=8\longrightarrow\mathcal{N}=2$ reduction originating $J_{3,M}^{\mathbb{R}}$ truncates away:

1) 14 $m=0$ and 14 $m\neq0$ vectors' scalar degrees of freedom, both sets sitting in the $(\mathbf{14},\mathbf{1})$ of $USp(6)\otimes SU(2)_H$;

2) 12 $m\neq0$ hypers' scalar degrees of freedom, sitting in the $(\mathbf{6},\mathbf{2})$ of $USp(6)\otimes SU(2)_H$.

The resulting $\mathcal{N}=2$ $J_{3,M}^{\mathbb{R}}$ spectrum is composed by 28 $m=0$ real hypers' scalar degrees of freedom (rearranging in 7 quaternionic hypermultiplet scalar), and by $n_V+1=2$ $m\neq0$ and $n_V-1=0$ $m=0$ real vectors' scalar degrees of freedom, whose mass degeneracy pattern thus confirms once again the Hessian splitting found in [10] (no "flat" directions of non-BPS $Z\neq0$ Hessian, implying that the non-BPS $Z\neq0$ critical points of $V_{BH,\mathcal{N}=2}$ in the model $J_{3,M}^{\mathbb{R}}$ are *all* stable).

For what concerns the other "*mirror*" models, there is nothing more to say. Indeed, $J_{3,M}^{\mathbb{C}}$ has $n_V=0$ and thus it corresponds to a Reissner-Nördstrom (extremal) BH with (graviphoton) charges p^0 and q_0, only admitting $\frac{1}{2}$-BPS critical points for $V_{BH,\mathcal{N}=2}$. Furthermore, as previously mentioned, $J_{3,M}^{\mathbb{H}}$ does not exist (*at least* as far $d=4$ is concerned), and *stu* is *self-mirror*: $stu_{,M}=stu$.

6 Conclusion

In the present paper, in order to understand more in depth the nature of the non-BPS solutions to attractor equations in $\mathcal{N}=8$, $d=4$ supergravity, we considered the supersymmetry reduction down to $\mathcal{N}=2$, $d=4$ magic supergravities (and their *"mirror"* theories). The multiplets' content is given by n_V vector supermultiplets, whose complex scalars span a special Kähler manifold of dimension n_V, and by n_H hypermultiplets, whose quaternionic scalars span a quaternionic Kähler manifold of dimension n_H.

The mass spectrum of vector multiplets' scalars (the only relevant for the Attractor Mechanism in ungauged supergravities) in $\mathcal{N}=2$ magic supergravities has been studied in [21]. By taking into account also the "hidden" modes truncated away in the supersymmetry reduction $\mathcal{N}=8\longrightarrow\mathcal{N}=2$, the splittings of the $\mathcal{N}=2$ spectra should reproduce the splittings of the full spectra of the 70 real scalars of the parent $\mathcal{N}=8$ theory. We have shown how this works, and in particular we reproduced the result of [10] about the mass splitting of the modes of the $\mathcal{N}=2$ non-BPS $Z\neq0$ Hessian.

By the supersymmetry reduction $\mathcal{N} = 8 \longrightarrow \mathcal{N} = 2$, the eventual instability of $\mathcal{N} = 2$ non-BPS $Z \neq 0$ solutions to attractor equations studied in [10] should reflect in a possible instability of $\mathcal{N} = 8$ non-BPS critical points of V_{BH} in $\mathcal{N} = 8$, $d = 4$ supergravity.

On the other hand, by assuming that supersymmetry determines the $\mathcal{N} = 8$, $\frac{1}{8}$-BPS critical points to be stable, it is possible to argue that the $\mathcal{N} = 2$ non-BPS $Z = 0$ critical points of $V_{BH,\mathcal{N}=2}$ should be stable (beside the $\mathcal{N} = 2$, $\frac{1}{2}$-BPS critical points, whose stability is known after [5]). Correspondingly, when covariantly differentiating $V_{BH,\mathcal{N}=2}$ beyond the second order, the eventual "flat" directions of the non-BPS $Z = 0$ Hessian should suitably lift to directions with strictly positive eigenvalues, or remain "flat" at all orders. Among the considered models, only the $\mathcal{N} = 2$, $d = 4$ *stu* supergravity (having $(n_V, n_H) = (3, 4)$, and thus *self-mirror*) exhibit non-BPS, $Z = 0$ critical points stable already at the Hessian level. This can be understood by noticing that in such an $\mathcal{N} = 2$ framework *triality symmetry* puts non-BPS $Z = 0$ critical points on the very same footing of $\frac{1}{2}$-BPS critical points, which are always stable [5] and thus do not have any "flat" direction at all.

We conclude by saying that our analysis could be applied to non-BPS critical points of V_{BH} in $2 < \mathcal{N} < 8$, $(d = 4)$ extended supergravities, eventually comparing the $\mathcal{N} = 8$ non-BPS spectrum with spectra arising in $2 < \mathcal{N} < 8$ theories obtained by consistent supersymmetry reductions (along the lines of [37]), as done in [36] for the $\mathcal{N} = 8$, $\frac{1}{8}$-BPS spectrum. Ultimately, such a procedure could be performed for the $\mathcal{N} = 1$, $d = 4$ reduction of these theories, especially of the $\mathcal{N} = 2$ SK d-geometries [30].

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